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The 3-Sphere Regular Cellulation Conjecture

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Let X be a CW-complex [8] on the 3-sphere $S^3 = \{x \in R^4 : |x| = 1\}$ with its standard topology. X is also called a *cellulation*. The ascending sequence $X^0 \subset X^1 \subset X^2 \subset X^3 = X$ of closed subspaces of X satisfies the following conditions:

[1] X^0 is a discrete set of points (0-cells)

[2] For $0 < k \leq 3$, $X^k - X^{k-1}$ is the disjoint union of open subspaces, called k-cells, each of which homeomorphic to the open k-dimensional ball $U^k (= \{x \in R^k : |x| < 1\})$.

X is a *regular* cellulation if the boundary of every k-cell is homeomorphic to the $k-1$-dimensional sphere S^{k-1}, for $1 \leq k \leq 3$.

A biconnected graph $G = (V, E)$ is 2-connected if $|V| > 2$. The *3-sphere regular cellulation conjecture* claims that every 2-connected graph is the 1-dimensional skeleton of a regular cellulation of the 3-dimensional sphere [5]. The conjecture is trivially true for planar graphs. Biconnectivity is a necessary condition for a graph to satisfy such property. Therefore, the question would be equivalent to the biconnectivity test if the conjecture were proved to be true. On the contrary, it is not even clear whether such decision problem is computationally tractable. In [1], the conjecture was proved for hamiltonian graphs and complete graphs are hamiltonian. This consideration suggested that this property might hold when the graph lies, as far as embeddability into surfaces is concerned, in between a planar one and a complete one. Moreover, such extremal results were obtained for k-partite graphs since complete k-partite graphs verify the conjecture for every k [4]. Finally, a superclass of planar graphs and complete k-partite graphs verifying the conjecture was introduced in [6] and called the class of *extended split* graphs. Extended split graphs are, therefore, the state of the art for the proof of this open problem.

The 3-sphere regular cellulation conjecture was given for graphs with at least two cycles in [3] because we assumed that two 2-cells cannot share the same boundary in order to relate it to the concept of spatiality degree [7]. The *spatiality degree* of a connected graph G is the maximum number of 3-cells that the cellulation of a 3-sphere can have with G as a 1-dimensional skeleton, assuming that two distinct 2-cells of the complex cannot share the same boundary and the 2-dimensional skeleton is regular. In [1], [2], it is shown that the 3-sphere regular cellulation conjecture is true if and only if the spatiality degree of a 2-connected graph $G = (V, E)$ with at least two cycles is equal to $2(|E| - |V|)$. We denote the spatiality degree of a connected graph G with $s(G)$. In [1], it is also shown that, for any connected graph G, $s(G) = \sum_{i=1}^{k} s(B_i) - k + 1$ where $B_1 \cdots B_k$ are the biconnected components of G. It follows that computing the spatiality degree would be an interesting problem only if the conjecture were false.

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