Formal Methods in Software Development

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Symbolic Model Checking

Basic Idea: to represent Kripke structures by using boolean functions:

1. sets of states (as well as relations) are represented by their characteristic function: $x \in S \Leftrightarrow c_S(x)$=True
2. Model Checking problems (such as reachability) solved by working **on set of states**, manipulating their charactheristic functions.
3. Set of states satisfying a formula of a temporal logic are characterized as the fixpoint of a monotone operator.

All this works (sometimes!) thanks to an efficient tool to manipulate boolean functions: **Ordered Binary Decision Diagrams** (OBDDs).

Lesson 7a:

Ordered Binary Decision Diagrams (OBDDs)

Binary Decision Trees

A **binary decision tree** is a ***rooted, directed*** binary tree that contains two types of vertices:

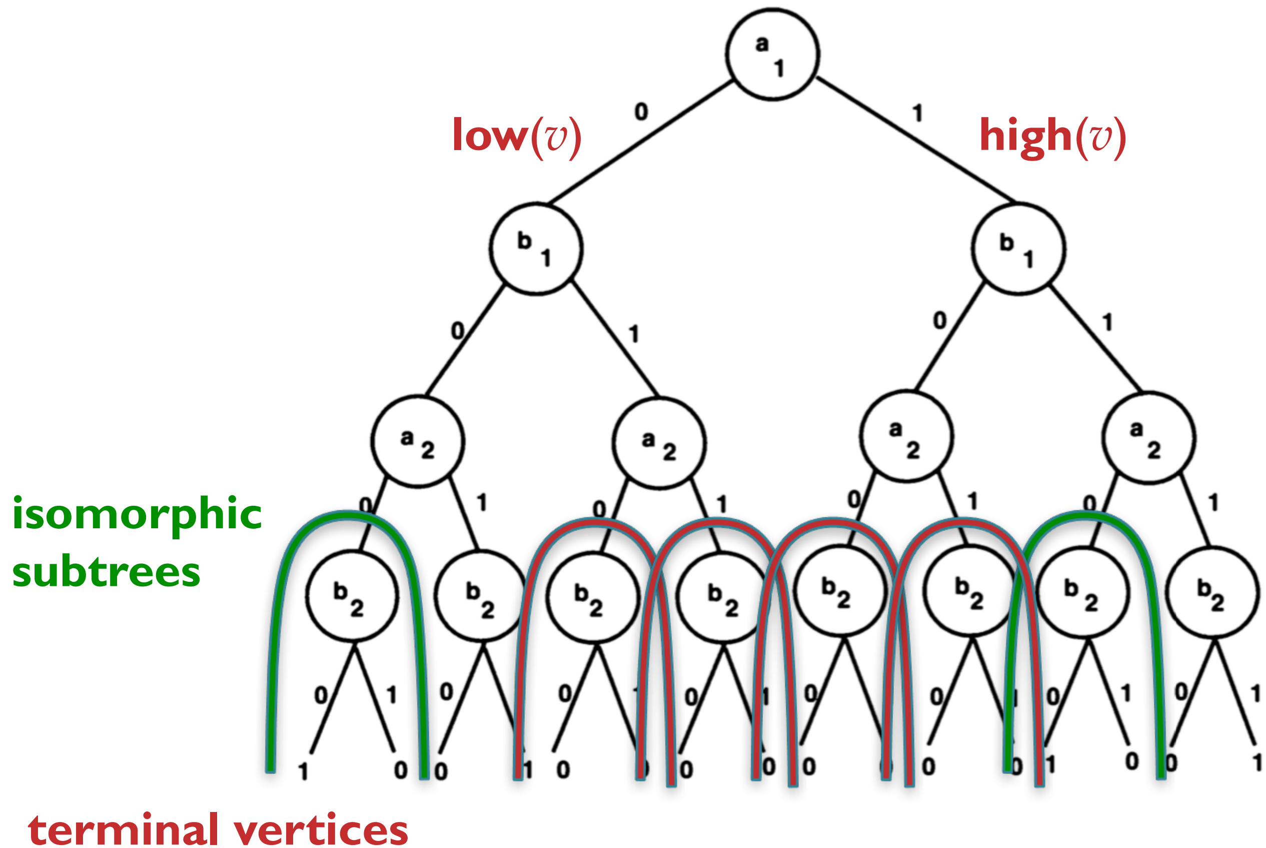
- **non-terminal** vertices v labeled by a variable var(v) and successors low(v) (when e v has value 0) and high(v) (when v has value 1).
- a **terminal** vertex v labeled by value(v) that is either 0 or 1.

A binary decision tree represents a boolean function.

Each path represents an assignment to boolean variables and the value of the terminal node represents the value of the function for that assignment.

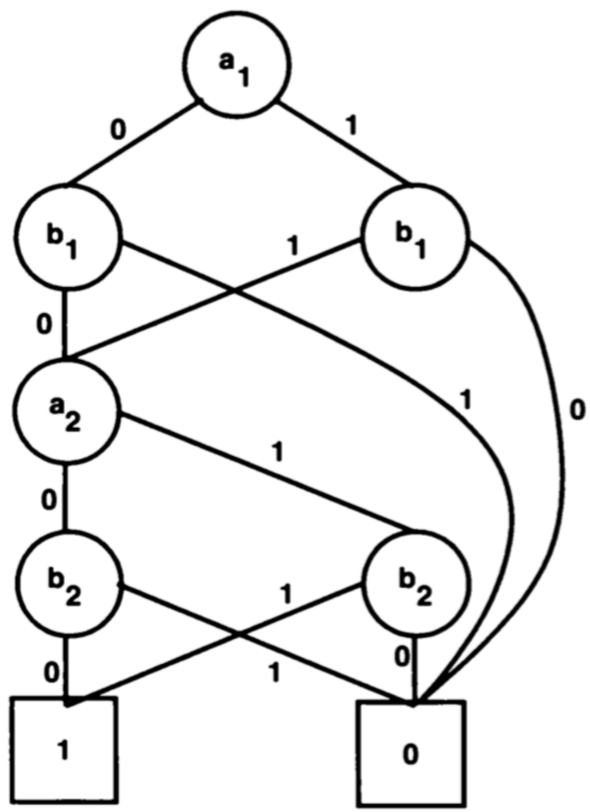
Binary Decision Trees

Example: two bit comparator: $f(a_1, a_2, b_1, b_2) = a_1 \leftrightarrow b_1 \wedge a_2 \leftrightarrow b_2$



Binary Decision Diagrams

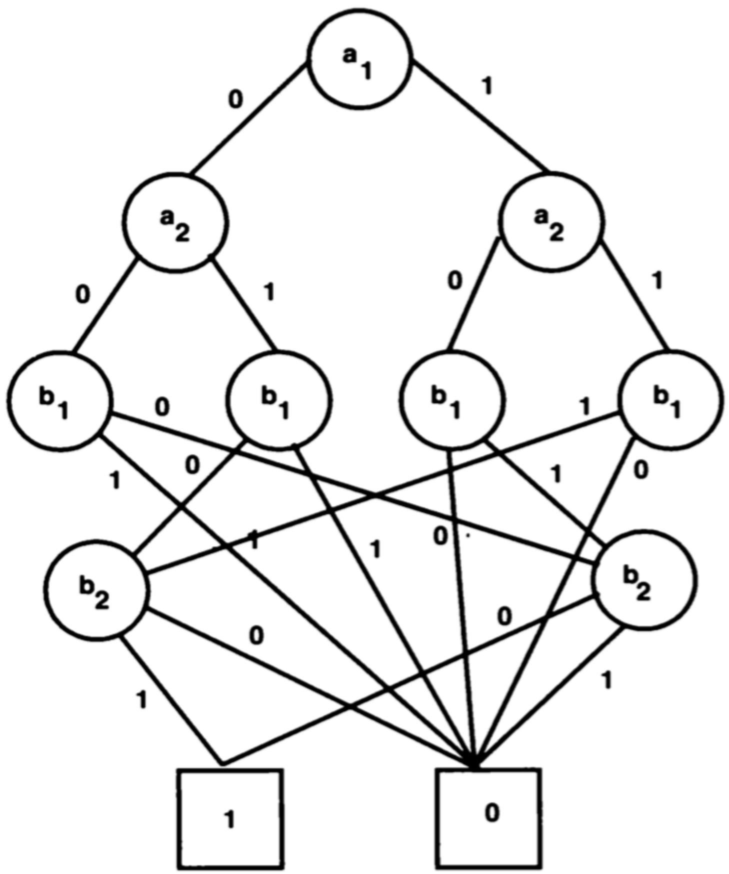
Idea: merging isomorphic subtrees (trivially, for example, just two terminal nodes): this leads to Directed Acyclic Graphs (DAGs)



Example: two bit comparator:
$f(a_1, a_2, b_1, b_2) = a_1 \leftrightarrow b_1 \land a_2 \leftrightarrow b_2$

Binary Decision Diagrams

Idea: The Binary Decision Diagram is highly dependent from the order of variables.



Example: two bit comparator:
$f(a_1, a_2, b_1, b_2) = a_1 \leftrightarrow b_1 \wedge a_2 \leftrightarrow b_2$
with the order $a_1 < a_2 < b_1 < b_2$

In the n bit comparator, using this order, the BDD has $3 \cdot 2^n$-1

With the order $a_1 < b_1 < \ldots < a_i < b_i < \ldots < a_n < b_n$ it has $3n$+2 nodes

Canonical Forms

For several applications it is desirable to have a canonical form for BDDs.

Definition: Two BDDs B_1 and B_2 are **isomorphic** if there exists a bijection $h : V(B_1) \to V(B_2)$ that maps terminal to terminal and non-terminal to non-terminal such that: value(v)=value($h(v)$), h(low(v))=low($h(v)$), and h(high(v))=high($h(v)$).

Canonical representations can be obtained by:

1. Imposing a total ordering on variables: if u has a successor v, then var(u)<var(v)
2. Avoid isomorphic sub-trees or redundant vertices.

Condition 2. can be obtained by using a reduce function that is linear in the size of the DAG.

Reduction to a canonical form

Remove duplicate terminals: eliminate all but one terminal vertex with a given label and redirect all arcs to eliminated vertices to the remaining one.

Remove duplicate nonterminals: If there exist two nonterminal u and v such that var(u)=var(v), low(u)=low(v), and high(u)=high(v) then eliminate one of them and redirect all incoming arcs to the remaining vertex.

Remove redundant tests: if low(u)=high(u) then eliminate the vertex u and redirect incoming arcs to low(u) [=high(u)].

This procedure can be implemented bottom-up, linear in the size of the BDD.

Some consequences:

- Checking equivalence of boolean functions corresponds to checking OBDDs isomorphisms.
- SAT on OBDDs is just checking if it is (not) the trivial OBDD.

Negative Results about OBDDs

- It is **NP-complete** to find the variable ordering for a boolean function $f(x_1, \ldots, x_n)$ **that makes** the size of the **OBDD** representing f **optimal**.
- There are boolean functions $f(x_1, \ldots, x_n)$ such that **the OBDD is exponential in *n* for all variable orders**. For example, the mid bit of the n bit product.

However, several heuristics give good results: for example, **related variables should be close in the ordering** (as in the n bit comparator example)

OBDD packages usually use **dynamic reordering** when heuristic seems to fail.

Logical operators using OBDDs

Restriction: $f|_{x_i=b}(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i\text{-}1}, b, x_{i+1}, \ldots x_n)$

Just find the node w such that $\mathsf{var}(w) = x_{i\text{-}1}$, remove such node and replace all arcs $v \rightarrow w$ with $v \rightarrow \mathsf{low}(w)$ if b=0 and with $v \rightarrow \mathsf{high}(w)$ if b=1. The resulting OBDD may not be in canonical form, so reduce must apply to it.

Using restriction, one can easily compute **Shannon expansion**:

$f = (\neg x \wedge f|_{x=0}) \vee (x \wedge f|_{x=1})$

Then binary operations can be recursively computed stemming from Shannon expansion as follows.

To simplify notations:

* v, v' are the root of OBDDs representing f and f'.
* x, x' are $\mathsf{var}(v)$ and $\mathsf{var}(v')$

Logical operators using OBDDs

Let $\star$ be an arbitrary two-argument boolean connective.

- If v and v' are both terminals, $f \star f'$=value(v) $\star$ value(v')
- If x=x' then using Shannon expansion, we have:

$$f \star f' = (\neg x \wedge (f|_{x=0} \star f'|_{x=0})) \vee (x \wedge (f|_{x=1} \star f'|_{x=1}))$$

  The root of the OBDD is w such that var(w)=x and low(w)=($f|_{x=0} \star f'|_{x=0}$) and high(w)=($f|_{x=1} \star f'|_{x=1}$)

- If x<x' then $f'|_{x=0} = f'|_{x=1} = f'$ since f' does not depend on x'. In this case Shannon expansion simplifies to:

$$f \star f' = (\neg x \wedge (f|_{x=0} \star f')) \vee (x \wedge (f|_{x=1} \star f'))$$

  and the OBDD is computed as before with x as root.

- If x'<x: symmetric to the previous case.

Since **each problem generates two subproblems**, to prevent exponential behaviour **dynamic programming** must be used! $\mathcal{O}(|f| \cdot |f'|)$.

Some optimizations

Negation can be computed just flipping terminal nodes.

A single multirooted DAG can be used to represent several boolean functions that share subgraphs. In this case, *f* and *f′* are the same if they have the same root!

Adding label to edges to denote boolean negation. In this case *f* and ¬*f* can be represented by a single OBDD.

OBDDs can be viewed as a DFA. A *n*-ary boolean function can be seen as the set of string *x* in $\{0, 1\}^n$ such that *f(x)* = 1. The minimal automata that accept this language is an alternative canonical form for *f*.

Standard boolean connectives can be seen as operation between languages (for example $\wedge$ is intersection etc.)

Lesson 7b:

Using OBDDs to represent Kripke Structures

Characteristic functions

Let Q be a n-ary relation over {0,1}. Q can be represented by its characteristic function: $(x_1, \ldots, x_n) \in Q$ iff $f_Q (x_1, \ldots, x_n)=1$.

Let Q be a n-ary relation over a domain D. For simplicity we assume that $|D|=2^m$ for some $m>1$. Elements of D can be encoded using a bijection $\phi : \{0,1\}^m \rightarrow D$. Q can be represented by a $m \times n$-ary boolean characteristic function according to $f_Q(d_1, \ldots, d_n)=1$ iff $Q(\phi(d_1), \ldots, \phi(d_n))$ where $d_1, \ldots, d_n$ are vectors of length m of boolean variables.

Sets can be viewed simply as unary relations.

Kripke structures as OBDDs

Let $\mathcal{M}$=(S, R, L) be a Kripke structure.

An encoding function ϕ : encodes states of S.

S is the constant function 1 on $\{0,1\}^m$. Subsets of S are represented by their characteristic functions.

R is represented by a characteristic function $f_R : \{0,1\}^{2m} \rightarrow \{0,1\}$.

The mapping L is represented by an OBDD f_p for each atomic proposition p, such that f_p is the characteristic function of the set $\{ s \in S \mid p \in L(s)\}$.

Along the same lines, one can represent the set of initial states I or a set of (unconditionally) fairness constraints F=$\{P_1, \ldots, P_k\}$.

Of course, $\mathcal{M}$ is not explicitely generated and then converted in its representation via OBDD, but rather OBDDs are generated starting from a high level description of $\mathcal{M}$ (for example programs!)

Lesson 7c:

Fixpoints

Classical FixPoint Theorem

Fixpoints has a relevant role in Logic and Theoretical Computer Science. For example, they are used to define semantics of Programming Languages (recursive definitions) or equivalences (bisimulation).

Given an function $T: L \rightarrow L$, $x \in L$ is a **fixpoint** of T if T(x)=x. μT (resp. υT) denotes the minimum (resp. maximum) fixpoint of T.

Definition: A **complete lattice** $(L, \leq)$ is a partially ordered set, such that each subset $A \subseteq L$ has a greatest lower bound $\sqcap A$ (**glb** or **inf** standing for infimum) and a least upper bound $\sqcup A$ (**lub** or **sup** standing for supremum).

$$\sup A = \min\{x \mid \forall a \in A. x \geq a\} \quad \text{and} \quad \inf A = \max\{x \mid \forall a \in A. x \leq a\}$$

Example: Given a set S, $(P(S), \subseteq)$ is a complete lattice where if A $\subseteq P(S)$ then inf $A = \cap_{a \in A}\, a$ and sup $A = \cup_{a \in A}\, a$.

Observation: a complete lattice L has always a minimum, that is $\bot = \inf \varnothing$ and a maximum $\top = \sup L$. This implies that $L \neq \varnothing$.

Knarster-Tarskij Theorem

Def: T: $L \rightarrow L$ is **monotonic** if $x \leq y$ implies $T(x) \leq T(y)$.

Theorem: [KNARSTER-TARSKIJ] If L is a complete lattice and $T : L \rightarrow L$ is *monotonic,* then T has a minimum fixpoint μT and a maximum fixpoint υT.

Moreover, $\mu T = \inf \{x \mid T(x) \leq x\}$, and $\upsilon T = \sup \{x \mid x \leq T(x)\}$.

Proof: Let $G = \{x \mid T(x) \leq x\}$ and g=inf G. We first show that $g \in G$. $g \leq x$, $\forall x \in G$. By monocity of T, $T(g) \leq T(x) \leq x$. But, being an inf, g is the maximum lower bound, therefore $T(g) \leq g$. Hence $g \in G$.

From $T(g) \leq g$ we have $T(T(g)) \leq T(g)$. But this implies that $T(g) \in G$. Therefore $g \leq T(g)$. And therefore $g = T(g)$, that is g is a fixpoint.

Finally, let g'=inf$\{x \mid T(x)=x\}$. Since g is a fixpoint $g' \leq g$. But since $\{x \mid T(x)=x\} \subseteq \{x \mid T(x) \leq x\}$, we have also $g \leq g'$. Hence g is the minimum fixpoint of T. A dual argument works for υT. $\Box$

Knarster-Tarskij Theorem II

Definition: Let $T : L \rightarrow L$, we define **transfinite powers** of T as follows: $T^0 = \bot$. $T^{\alpha+1} = T(T^\alpha)$ and if λ is a limit ordinal, $T^\lambda = \sup_{\alpha<\lambda} T^\alpha$. Dually, we define the **transfinite downward powers** as follows: $T_0 = \top$. $T_{\alpha+1} = T(T_\alpha)$ and if λ is a limit ordinal, $T_\lambda = \inf_{\alpha<\lambda} T_\alpha$.

Theorem: If L is a complete lattice and $T : L \rightarrow L$ is *monotonic*, then $T^\alpha \leq \mu T$ and $\upsilon T \leq T_\alpha$. Moreover, there exist two ordinals β_1 and β_2 such that $\mu T = T^\alpha$ for all $\alpha \geq \beta_1$ and $\upsilon T = T_\alpha$ for all $\alpha \geq \beta_2$.

Proof: 1) $T^\alpha \leq \mu T$. Trivially, $T^0 = \bot \leq \mu T$. If $T^\alpha \leq \mu T$, by monotonicity of T we have $T(T^\alpha) \leq T(\mu T)$ that means $T^{\alpha+1} \leq \mu T$. $T^\lambda = \sup_{\alpha<\lambda} T^\alpha$ and all $T^\alpha \leq \mu T$ (by transfinite inductive hypothesis) we have the thesis because limits preserve ≤.

2) $T^\alpha \leq T^{\alpha+1}$. Trivially, $T^0 = \bot \leq T^1$. Assuming $T^\alpha \leq T^{\alpha+1}$, by monotonicity, we have $T(T^\alpha) \leq T(T^{\alpha+1})$, hence $T^{\alpha+1} \leq T^{\alpha+2}$.

3) If $\alpha \leq \beta$ then $T^\alpha \leq T^\beta$. The property is trivial using 2) and observing again that limits preserve ≤ (for limit ordinals).

Knarster-Tarskij Theorem II

Proof: (cntnd)

4) If $\alpha \leq \beta$ and $T^\alpha = T^\beta$ then $T^\alpha = \mu T$. If $T^\alpha = T^\beta$ and since by 2) the sequence T^α is ascending ordered, all $T^\alpha = T^\gamma = T^\beta$ for all γ such that $\alpha \leq \gamma \leq \beta$. But this implies that $T^\alpha = T(T^\alpha)$, that is T^α is a fixpoint and by 1) T^α is μT.

5) There exists α such that $T^\alpha = \mu T$. By contradiction, assume that this is not the case. By 2) and 4) this implies that the sequence of powers of T is **strictly** ordered and contains **distinct** elements, defining an injection from the set of ordinals into L. Absurd. (for any set L, there is an ordinal of "bigger" cardinality).

All these reasoning works dually for downward powers and for υT. □

Set Operators

Given a set S, the powerset $\mathcal{P}(S)$ is a complete lattice, ordered by $\subseteq$ (as expected sup is $\cup$ and inf is $\cap$).

A function $\tau : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ is called a **predicate transformer**.

- τ is **monotonic** if $Q \subseteq Q'$ implies $\tau(Q) \subseteq \tau(Q')$
- τ is **finitary** if $x \in \tau(Q)$ if and only if $\exists\, Q_0 \subseteq Q$, Q_0 finite such that $x \in \tau(Q_0)$ implies $\tau(Q) \subseteq \tau(Q')$
- τ is $\cup$ **-continuous** if for a sequence $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \ldots$ we have that $\cup_i \tau(Q_i) = \tau(\cup_i Q_i)$
- τ is $\cap$ **-continuous** if for a sequence $Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq \ldots$ we have that $\cap_i \tau(Q_i) = \tau(\cap_i Q_i)$

Finitary Operators

Theorem: $\tau : \mathcal{P}(S) \to \mathcal{P}(S)$ is $\cup$-continuous if and only if it τ is monotonic and finitary.

Proof: by the following two lemmas. □

Lemma: If τ is monotone and $\{x_j\}_{j\in J}$ is an ascending chain then $\sup_{j\in J} \tau(x_j) \subseteq \tau(\sup_{j\in J} x_j)$

Proof: for all $i \in J$, we have: $x_i \subseteq \sup_{j\in J} x_j$. By monotonicity of τ we have that $\tau(x_i) \subseteq \tau(\sup_{j\in J} x_j)$, that is, $\tau(\sup_{j\in J} x_j)$ is an upper bound of the chain $\{\tau(x_i)\}_{i\in J}$. Being sup $\{\tau(x_i)\}_{i\in J}$ the minimum of upper bounds, we get the thesis. □

Lemma: If τ is monotone and finitary and $\{x_j\}_{j\in J}$ is an ascending chain then $\sup_{j\in J} \tau(x_j) \supseteq \tau(\sup_{j\in J} x_j)$.

Proof: Let $y \in \tau(\sup_{j\in J} x_j)$. There exists a finite set $z_0 \subseteq \sup_{j\in J} x_j$ such that $y \in \tau(z_0)$. Since $\{x_j\}_{j\in J}$ is a chain, there exists k such that $z_0 \subseteq x_k \subseteq \sup_{j\in J} x_j$. Therefore y $\in \sup_{j\in J} \tau(x_j)$. □

Kleene Fixpoint Theorem

Theorem: [KLEENE] If τ is $\cup$-continuous then $\mu\tau = \bigcup_{i\in N} \tau^i(\varnothing) = \tau^\omega$.

Proof:

$\tau^\omega = \sup \{\tau^i \mid i < \omega\}$	(def. of τ^ω)	
$= \sup \{\tau(\tau^i) \mid i < \omega\}$	(prestige ☺)	
$= \tau(\sup \{\tau^i \mid i < \omega\})$	(continuity)	
$= \tau(\tau^\omega)$	(def. of τ^ω)	□

Question: Which is the dual notion of finitary? That ensure that for a monotonic operator τ we have $\upsilon\tau = \bigcap_{i\in N} \tau_i(\varnothing) = \tau_\omega$?

For the purposes of Model Checking… all this is a bit too much. In a finite S, monotonic the chain $\tau^i(\varnothing)$ reaches the least fixpoint $\mu\tau$ after a finite number of steps and $\tau_i(S)$ reaches the greatest fixpoint $\upsilon\tau$ after a finite number of steps! On a finite set, a monotonic operator is necessarily finitary!

Finite Fixpoint properties

Proofs of the following three lemmas can be easily obtained specializing proofs of the general case to the finite case.

Lemma: If S is finite and τ is monotonic, then τ is also $\cup$-continuous and $\cap$-continuous.

Lemma: If S is finite and τ is monotonic, then there exist two integers p, q such that $\tau^p(\varnothing) = \tau^i(\varnothing)$ and $\tau_q(S) = \tau_j(S)$ for all $i \geq p$ and for all $j \geq q$.

Lemma: If S is finite and τ is monotonic, then there exist two integers p and q such that $\tau^p(\varnothing) = \mu\tau$ and $\tau_q(S) = \upsilon\tau$.

Computing Finite Fixpoints - Lfp

Using characteristic functions for representing sets, the constant False is the emptyset. The invariant of the following program is $Q' \subseteq \mu\tau \wedge Q' = \tau(Q)$. It terminates because the guard of the while implies $Q \neq Q'$. The computed sequence is **strictly increasing** (wrt $\subseteq$) and hence $|S|$ is an upperbound to the number of iterations.

```
function Lfp(Tau : PredicateTransformer) : Predicate
    Q := False;
    Q' := Tau(Q);
    while (Q ≠ Q') do
        Q := Q';
        Q' := Tau(Q');
    end while;
    return(Q);
end function
```

Computing Finite Fixpoints - Gfp

Using characteristic functions for representing sets, the constant True is the set S. The invariant of the following program is $Q' \supseteq \upsilon\tau \wedge Q' = \tau(Q)$. It terminates because the guard of the while implies $Q \neq Q'$. The computed sequence is **strictly decreasing** (wrt $\subseteq$) and hence $|S|$ is an upperbound to the number of iterations.

```
function Gfp(Tau : PredicateTransformer) : Predicate
    Q := True;
    Q' := Tau(Q);
    while (Q ≠ Q') do
        Q := Q';
        Q' := Tau(Q');
    end while;
    return(Q);
end function
```

Lesson 7d:

Symbolic CTL model checking

FixPoints and CTL

Identifying a formula *f* with the set of states Sat(*f*)={*s* | $\mathcal{M}$, *s* ⊨ f },
we can characterize temporal operators as predicate transformers and their semantics as fixpoints of such operators.

Intuitively: **eventualities** (**F** and **U**) are **least fixpoints** and properties that **hold forever** (**R** and **G**) are **greatest fixpoints**.

They use **expansion laws** for **F**, **U**, **R**, and **G** using **X**.

- $\mathbf{AF}\ f_1 = \mu Z\ .\ f_1 \vee \mathbf{AX}\ Z$
- $\mathbf{EF}\ f_1 = \mu Z\ .\ f_1 \vee \mathbf{EX}\ Z$
- $\mathbf{AG}\ f_1 = \nu Z\ .\ f_1 \wedge \mathbf{AX}\ Z$
- $\mathbf{EG}\ f_1 = \nu Z\ .\ f_1 \wedge \mathbf{EX}\ Z$
- $\mathbf{A}[f_1\ \mathbf{U}\ f_2] = \mu Z\ .\ f_2 \vee (f_1 \wedge \mathbf{AX}\ Z)$
- $\mathbf{E}[f_1\ \mathbf{U}\ f_2] = \mu Z\ .\ f_2 \vee (f_1 \wedge \mathbf{EX}\ Z)$
- $\mathbf{A}[f_1\ \mathbf{R}\ f_2] = \nu Z\ .\ f_2 \wedge (f_1 \vee \mathbf{AX}\ Z)$
- $\mathbf{E}[f_1\ \mathbf{R}\ f_2] = \nu Z\ .\ f_2 \wedge (f_1 \vee \mathbf{EX}\ Z)$

EG *as a greatest fixpoint*

Lemma: The predicate transformer $\tau(Z)=f \wedge \mathbf{EX}\ Z$ is monotonic.

Proof: Let $Z_1 \subseteq Z_2$. Let $s \in \tau(Z_1)$. Then $s \vDash f$ and there exist a successor s' of s, such that $s' \in Z_1$. But then $s' \in Z_2$ and this implies that $s \in \tau(Z_2)$. □

Lemma: **EG** f is a fixpoint of the predicate transformer $\tau(Z)=f \wedge \mathbf{EX}\ Z$.

Proof: Suppose $s_0 \vDash \mathbf{EG}\ f$. Then there exists an infinite path $\pi = s_0 s_1 s_2 \ldots$ such that for all k, $s_k \vDash f$. This implies that $s_0 \vDash \mathbf{EG}\ f$ and $s_1 \vDash \mathbf{EG}\ f$, that is $s_0 \vDash \mathbf{EX\ EG}\ f$. Thus Sat(**EG** f) $\subseteq$ Sat($f \wedge$ **EX EG** f). Clearly Sat($f \wedge$ **EX EG** f) $\subseteq$ Sat(**EG** f) and hence they are equal. □

Lemma: **EG** f is the greatest fixpoint of the predicate transformer $\tau(Z)=f \wedge \mathbf{EX}\ Z$.

Proof: Being a fixpoint, $\mathbf{EG}\ f \subseteq \nu Z.\ f \wedge \mathbf{EX}\ Z = \cap_k \tau_k(S)$ for some k. Let $s \in \cap_k \tau_k(S)$. Since it is a fixpoint, $s \in \tau(\cap_k \tau_k(S))$. This implies that $s \vDash f$ and $\exists s'.R(s, s')$ and $s' \in \cap_k \tau_k(S)$. Applying this argument to s' we find an infinite sequence of states that belong to $\cap_k \tau_k(S)$ starting in s and thus $s \in \mathbf{EG}\ f$. □

EU *as a least fixpoint*

Lemma: The operator $\tau(Z) = f_2 \vee (f_1 \wedge \mathbf{EX}\ Z)$ is monotonic.

Proof: Let $Z_1 \subseteq Z_2$. Let $s \in \tau(Z_1)$. If $s \models f_2$ then $s \in \tau(Z_2)$. Otherwise $s \models f_1$ and there exists a successor s' of s such that $s' \in Z_1$. Since $s' \in Z_2$, we have also that $s \in \tau(Z_2)$. □

Lemma: $\mathbf{E}\ [f_1\ \mathbf{U}\ f_2]$ is a fixpoint of $\tau(Z) = f_2 \vee (f_1 \wedge \mathbf{EX}\ Z)$.

Proof: We have to show that $\mathrm{Sat}(\mathbf{E}\ [f_1\ \mathbf{U}\ f_2]) = \mathrm{Sat}(f_2 \vee (f_1 \wedge \mathbf{EX}\ \mathbf{E}\ [f_1\ \mathbf{U}\ f_2]))$. $s_0 \in \mathrm{Sat}(\mathbf{E}\ [f_1\ \mathbf{U}\ f_2])$ if and only there exists a path of length $k \geq 0$ such that $s_k \models f_2$ and $s_i \models f_1$ for $0 \leq i < k$ if and only if $s_0 \in \mathrm{Sat}(f_2 \vee (f_1 \wedge \mathbf{EX}\ \mathbf{E}\ [f_1\ \mathbf{U}\ f_2])$. □

Lemma: $\mathbf{E}\ [f_1\ \mathbf{U}\ f_2]$ is the least fixpoint of the predicate transformer $\tau(Z) = f_2 \vee (f_1 \wedge \mathbf{EX}\ Z)$.

Proof: Being a fixpoint, $\cup_i \tau_i(\varnothing) = \mu Z.\ f_2 \vee (f_1 \wedge \mathbf{EX}\ Z) \subseteq \mathbf{E}\ [f_1\ \mathbf{U}\ f_2]$. Let $s \in \mathbf{E}\ [f_1\ \mathbf{U}\ f_2]$. If $s \models f_2$ then $s \in \tau(Z)$ for any Z and so $s \in \cup_i \tau_i(\varnothing)$. Otherwise $s \models f_1$ and there exists a path of length $k \geq 0$ such that $s_k \models f_2$ and $s_j \models f_1$ for $0 \leq j < k$. It is easy to see that $s \in \tau_k(\varnothing)$. To be formal, induction on k. □

Exercises

Exercise 1: Which is the least fixpoint of the operator $\tau(Z)=f \wedge \mathbf{EX}\ Z$?

Exercise 2: Find a Kripke structure in which the greatest fixpoint of the operator $\tau(Z) = f_2 \vee (f_1 \wedge \mathbf{EX}\ Z)$ contains at least a state s such that $s \nvDash \mathbf{E}\ [f_1\ \mathbf{U}\ f_2]$.

Exercise 3: Find a monotonic but not continuous operator (of couse you must deal with infinite sets!)

Exercise 4: Which is the dual notion of finitary? That ensure that for a monotonic operator τ we have $\upsilon\tau = \bigcap_{i\in \mathrm{N}} \tau_i(\varnothing) = \tau_\omega$?

CTL model checking

The problem is to find three functions such that:

$$check(\mathbf{EX}\ f) = checkEX(check(f))$$

$$check(\mathbf{E}[f\ \mathbf{U}\ g]) = checkEU(check(f), Check(g))$$

$$check(\mathbf{EG}\ f) = checkEG(check(f))$$

Observe that the parameter of *check* is a CTL formula φ, its result is an OBDD representing the set of states satisfying φ.

The parameters of *checkEX*, *checkEU,* and *checkEG* **are OBDDs**.

CTL model checking

- *checkEX*($f(v)$) is strighforward. It is equivalent to $\exists v'.f(v') \wedge R(v,v')$.
- *checkEU*($f_1(v), f_2(v)$) is based on the characterization of **EU** as the least fixpoint of the predicate transformer $\mu Z. f_2(v) \vee (f_1(v) \wedge$ **EX** $Z)$.

It is computed a converging sequence of states $Q_1, \ldots, Q_i, \ldots$ Having the OBDD for Q_i and those for $f_1(v)$ and $f_2(v)$ one can easily compute those for Q_{i+1}. Observe that checking $Q_i = Q_{i+1}$ is strighforward.

- *checkEG*($f(v)$) is based on the characterization of **EG** as the greatest fixpoint of the predicate transformer $\upsilon Z. f_1(v) \wedge$ **EX** Z.

Quantified Boolean Formulas

In the previous slides we use formulas such as: $\exists v'.f(v') \wedge R(v,v')$.

They are **quantified boolean formulas**: They are equivalent to propositional formulas, but they allow a more succint representation.

Semantics:

- $\sigma \models \exists v f$ iff $\sigma\langle v \leftarrow 0\rangle \models f$ or $\sigma\langle v \leftarrow 1\rangle \models f$, and
- $\sigma \models \forall v f$ iff $\sigma\langle v \leftarrow 0\rangle \models f$ and $\sigma\langle v \leftarrow 1\rangle \models f$.

They can be represented as OBDD using restriction:

- $\exists x f = f|_{x\leftarrow 0} \vee f|_{x\leftarrow 1}$
- $\forall x f = f|_{x\leftarrow 0} \wedge f|_{x\leftarrow 1}$

Lesson 6

That's all Folks…

…Questions?