Formal Methods in Software Development

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Lesson 4a:

Finite Automata and Specifications

(Non-Det.) Finite Automata

A **finite automata** $\mathcal{A}$ is a 5-tuple $(\Sigma, Q, \delta, Q_0, F)$ where:

- Σ is the finite input *alphabet*,
- Q is the finite set of *states*,
- $\delta \subseteq Q \times \Sigma \times Q$ is the *transition relation*,
- $Q_0 \subseteq Q$ is the set of *initial states*,
- $F \subseteq Q$ is the set of *accepting states*.

Let w be a word in Σ^* of length $|w| = n$.

A **run** over w is a finite sequence of states $q_0 q_1 \ldots q_n$ such that $q_0 \in Q_0$ is an initial state and $(q_i, w_{i+1}, q_{i+1}) \in \delta$ for all $1 \leq i < n$.

A run is **accepting** if $q_n \in F$.

The automaton $\mathcal{A}$ **accepts** w if **there exists** an accepting run over w.

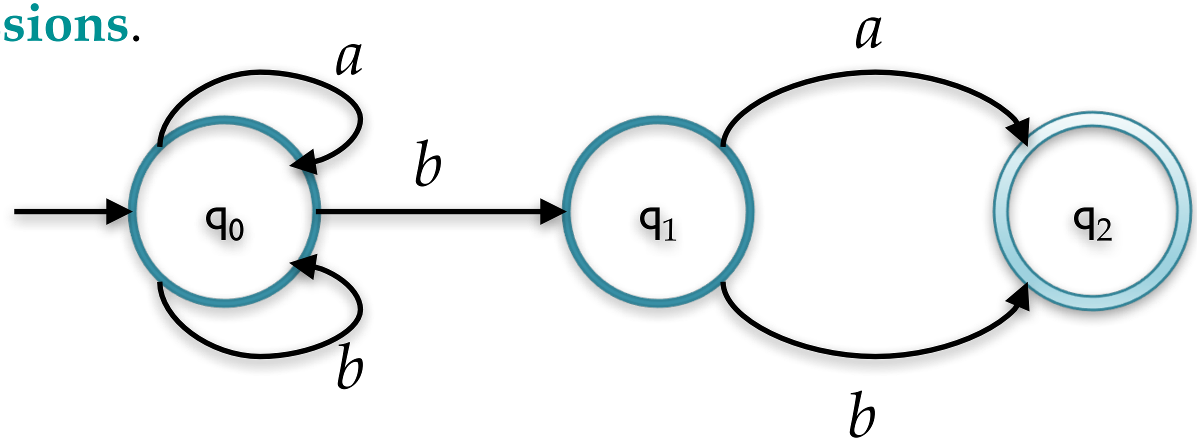
The **language** $\mathcal{L}(\mathcal{A}) \subseteq \Sigma^*$ consists of all the words accepted by $\mathcal{A}$.

(Non-Det.) Finite Automata

The automaton $\mathcal{A}$ is **deterministic** if δ is a function (for all states s and all symbols a there exists a unique next state $\delta(s, a)$ and a unique initial state ($|\delta(s, a)| \leq 1$ and $|Q_0| \leq 1$)).

For each non-deterministic automaton $\mathcal{A}$ there exists an automaton $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$. However, the size of $\mathcal{A}'$ **can be exponential** w.r.t. the size of $\mathcal{A}$.

The class of languages accepted by finite automata is the class of **regular languages**, that can be characterized by **regular expressions**.

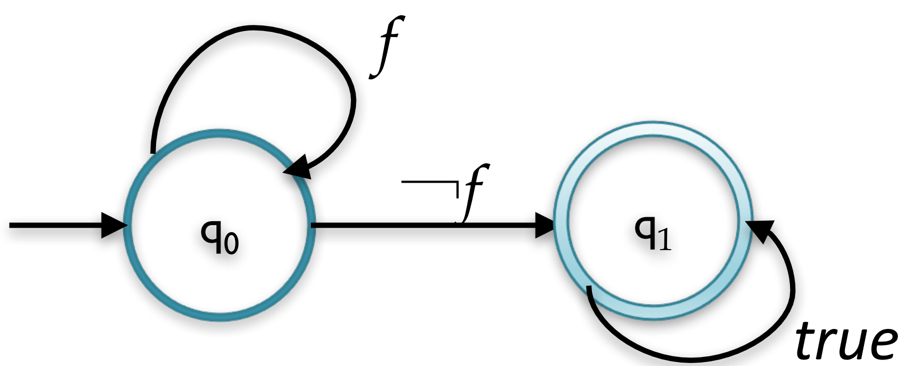


$(a+b)^*b(a+b)$

Regular Safety Properties

Definition. A safety property P is **regular** if its set of bad prefixes is a regular language over 2^{AP}.

Example: Every **invariant is a regular property**. Let f be the invariant property. The language of bad prefixes is $f^*(\neg f)\mathsf{true}^*$ (**we use a propositional formulas to identify subsets of *AP***).

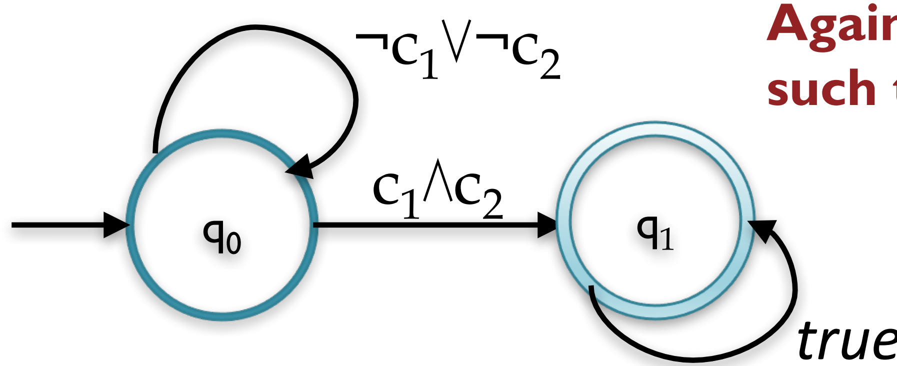


This automaton accepts words that **violates** the invariant f.

Remark: Here we assume f be a shorthand for $L(s) \vDash f$.

Regular Safety Properties

Example. The mutual exclusion property can be easily modeled by a NFA as follows.



Again: φ means $L(s)$ such that $L(s) \vDash \varphi$

This automaton accepts **minimal** bad prefixes of the mutual exclusion property.

Theorem. A safety property P is regular iff the set of **minimal** bad prefixes for P is regular.

Verifying Regular Safety Prop.

Idea. Run in parallel the system model $\mathcal{M}$ and the automaton for $\neg f$. $\mathcal{M} \vDash f$ **iff** $\text{traces}_{\text{fin}}(\mathcal{M}) \cap \text{badPrefixes}(P_f) = \varnothing$ **iff** $\text{traces}_{\text{fin}}(\mathcal{M}) \cap \mathcal{L}(\neg f) = \varnothing$.

Ingredients:

- build an automata for the intersection of two languages
- checking language emptiness

Definition: [**Product of a Transition System $\mathcal{M}$ and a NFA**]
Let $\mathcal{M} = (S, A, I, \rightarrow, AP, L)$ and $\mathcal{A} = (\Sigma, Q, \delta, Q_0, F)$, such that $\Sigma = 2^{AP}$ and $Q_0 \cap F = \varnothing$. Then $\mathcal{M} \otimes \mathcal{A} = (S', A, I', \rightarrow', AP', L')$ where:

- $S' = S \times Q$
- $(s, q) \rightarrow'_a (s', q')$ whenever $s \rightarrow_a s'$ and $\delta(q, L(s'), q')$
- $I' = \{(s, q) \mid s \in I \wedge \exists q_0 \in Q_0 \,.\, (q_0, L(s), q)\}$
- $AP' = Q$
- $L' : S \times Q \rightarrow 2^Q$ is given by $L'(s, q) = \{q\}$

This construction works also for **Kripke structures**.

Verifying Regular Safety Prop.

Let us define: $\neg F = P_{\text{inv}} = \bigwedge_{q_i \in F} \neg q_i$

Theorem. Let $\mathcal{A}$ be a NFA such that $\mathcal{L}(\mathcal{A})$ = badPrefixes(P) of some safety property P and let $\mathcal{M}$ be a transition system. Then the following are equivalent:

- $\mathcal{M} \vDash P$
- $\text{traces}_{\text{fin}}(\mathcal{M}) \cap \mathcal{L}(\mathcal{A}) = \varnothing$
- $\mathcal{M} \otimes \mathcal{A} \vDash P_{\text{inv}}$

Checking a **regular safety property** has been reduced to a **invariant checking**, that in turn it can be solved by a **reachability**.

Equivalently, **emptiness** of a regular language is a **reachability** problem (check whether accepting states are reachable from some initial state)

The accepted words are **counterexamples**

Lesson 4b:

Finite Automata Over Infinite Words

ω-regular Languages

ω-regular languages are a subset of infinite words Σ^ω over a finite alphabet Σ generated by ω-regular expressions.

Example: $(ab)^\omega = ababababab\ldots$ Observe that $(ab)^*$ is **an infinite set of finite** words, but $(ab)^\omega$ is a **single infinite word**.

The operator $^\omega$ lifts to languages. $\mathcal{L}^\omega = \{\, w_1 w_2 w_3 \ldots \mid w_i \in \mathcal{L} \,\}$

Definition: An ***ω*-regular expression** over Σ has the form:

$$G = E_1 \cdot F_1{}^\omega + \ldots + E_n \cdot F_n{}^\omega$$

where $n \geq 1$ and $E_1, F_1, \ldots, E_n\ F_n$ are regular expressions.

$$\mathcal{L}(\mathrm{G}) = \mathcal{L}(E_1) \cdot \mathcal{L}(F_1)^\omega \cup \ldots \cup \mathcal{L}(E_n) \cdot \mathcal{L}(F_n)^\omega$$

$\mathcal{L}$ is ω-regular if $\mathcal{L} = \mathcal{L}(\mathrm{G})$ for some ω-regular expression G.

ω-regular languages are closed under **union**, **intersection** and **complementation**.

Examples: $(a+b)^* \cdot b^\omega$ is the language of words with finitely many a's. $(b^* a)^\omega$ is the language of words with infinitely many a's.

(Non-Det.) Büchi Automata

A **non-determistic Büchi automata** $\mathcal{A}$ is a 5-tuple $(\Sigma, Q, \delta, Q_0, F)$ where:

This definition is exactly the same of NFA, but the semantics of accepted words change!

- Σ is the finite input *alphabet*,
- Q is the finite set of *states*,
- $\delta \subseteq Q \times \Sigma \times Q$ is the *transition relation*,
- $Q_0 \subseteq Q$ is the set of *initial states*,
- $F \subseteq Q$ is the set of *accepting states*.

Let w be an **infinite** word in Σ^ω. A **run** ρ over w is an **infinite** sequence of states $q_0 q_1 \ldots q_n \ldots$ such that $q_0 \in Q_0$ is an initial state and $(q_i, w_{i+1}, q_{i+1}) \in \delta$ for all $i \in \mathbb{N}$. $\inf(\rho)$ is the set of states that occur infinitely often in ρ.

A run is **accepting** if $q_j \in F$ for **infinitely many** i.

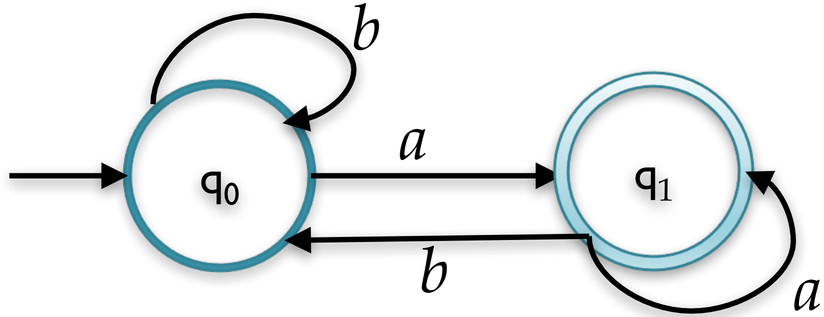
The automaton $\mathcal{A}$ **accepts** w if **there exists** an accepting run ρ over w such that $\inf(\rho) \cap \mathrm{F} \neq \varnothing$

The **language** $\mathcal{L}(\mathcal{A}) \subseteq \Sigma^\omega$ consists of all the words accepted by $\mathcal{A}$.

Büchi Autom. and ω-regular lang.

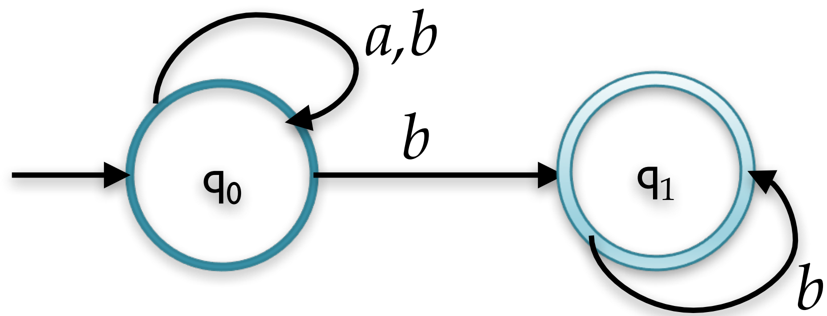
Theorem. The class of languages accepted by NBA's is exactly the class of ω-regular languages.

Examples: Infinitely many a's



deterministic

Finitely many a's



non-deterministic
b **in** q_0**: two trans.**

The automaton "knows" when the sequence of finitely many a's stops

NBA for $\mathcal{L}_1 + \mathcal{L}_2$ with $\mathcal{L}_1$, $\mathcal{L}_2$ ω-reg

Theorem. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are ω-regular, then $\mathcal{L}_1 \cup \mathcal{L}_2$ is ω-regular.

Proof: Given an automaton $\mathcal{A}_1 = (\Sigma, Q_1, \delta_1, I_1, F_1)$ accepting $\mathcal{L}_1$ and an automaton $\mathcal{A}_2 = (\Sigma, Q_2, \delta_2, I_2, F_2)$ accepting $\mathcal{L}_2$ we build the automata

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 = (\Sigma, Q_1 \cup Q_2, \delta, I_1 \cup I_2, F_1 \cup F_2)$$

where $(q, a, q') \in \delta$ if $(q, a, q') \in \delta_1$ or $(q, a, q') \in \delta_2$.

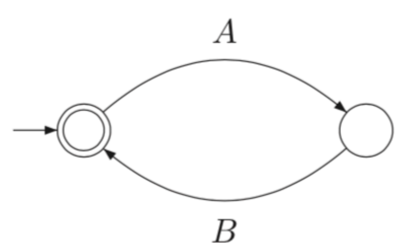
It is easy to see that $\mathcal{A}_1 + \mathcal{A}_{2\ \mathrm{a}}$ accepts $\mathcal{L}_1 \cup \mathcal{L}_2$. (**Exercise** ☺). □

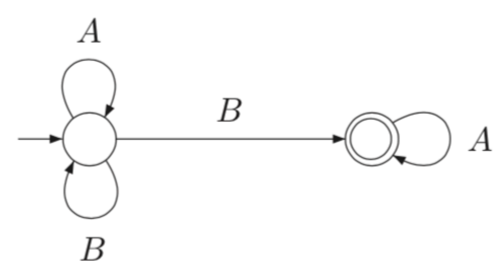
NBA for $\mathcal{L}_1 \cdot \mathcal{L}_2$, $\mathcal{L}_1$ reg. $\mathcal{L}_2$ ω-reg

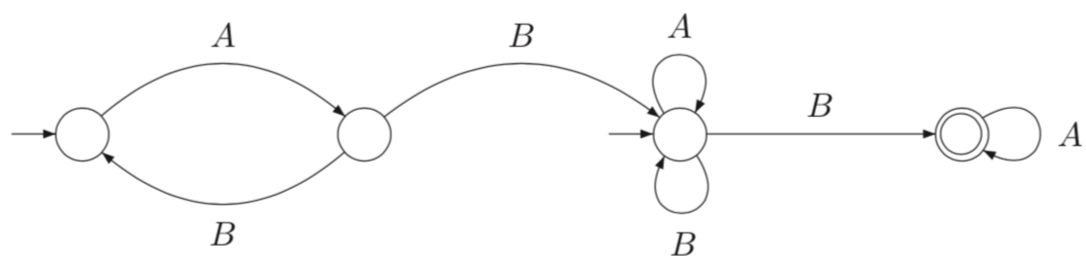
Taking the NFA $\mathcal{A}_1$ accepting $\mathcal{L}_1$, the basic trick is adding a transition to an initial state of the NBA $\mathcal{A}_2$ accepting $\mathcal{L}_2$, whenever there is a transition to a final state of $\mathcal{A}_1$. Observe that final states are those of the NBA $\mathcal{A}_2$. Observe that possible infinite runs inside $\mathcal{A}_1$ are not accepting.

Here an example with $\mathcal{L}_1=(ab)^*$, $\mathcal{L}_2=(a+b)^*ba^\omega$ and

$$\mathcal{L}_1 \cdot \mathcal{L}_2=(ab)^*(a+b)^*ba^\omega$$







NBA accepting $\mathcal{L}^\omega$, $\mathcal{L}$ regular

Insert a new initial (and accepting state) q_{new} and:
1. put a transition from q_{new} to any successor of initial states;
2. put a transition to q_{new} from any accepting state.

Show that the resulting NBA accept $\mathcal{L}^\omega$. **(Excercise ☺)**

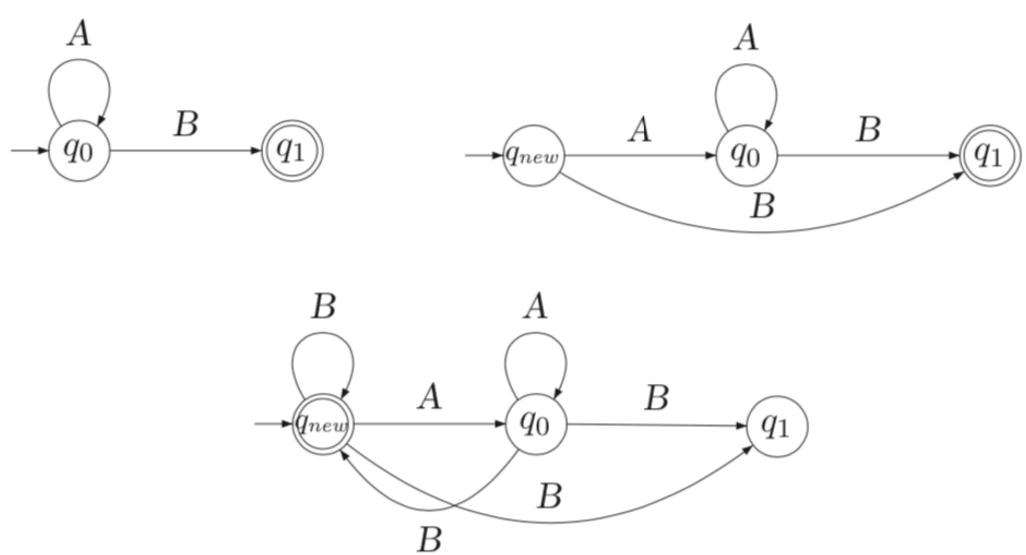


Figure 4.12: From an NFA accepting A^*B to an NBA accepting $(A^*B)^\omega$.

Non blocking NBA

As usual, we want automata with a total transition relation (**non-blocking**). If a computation **gets stuck**, it's not a problem for theory, it is just a **non-accepting computation** (the same for non-deterministic NFAs).

Proposition: For each NBA $\mathcal{A}$ there exists a non-blocking equivalnet NBA $\mathcal{A}'$ equivalent to $\mathcal{A}$.

Proof: Just add a sink (or trap) state q_{trap} and transitions to whenever a transition is not defined in some state.$\square$

More or less, the same trick works for Kripke structures and NFAs.

Remark: $\mathcal{A}$ **equivalent** to $\mathcal{A}'$ means that $\mathcal{L}(\mathcal{A})= \mathcal{L}(\mathcal{A}')$

Büchi Autom.: non-determinism

Theorem. There is **no** non-deterministic Büchi automata that accept the language $(a+b)^* b^\omega$.

Proof: Assume that there exists such automaton.

The word b^ω belongs to the language. There exists an accepting state q_1 such that $\delta^*(q_0, b^{n_1}) = q_1$ (δ is a function!).

The word $b^{n_1} a b^\omega$ belongs to the language. There exists an accepting state q_2 such that $\delta^*(q_0, b^{n_1} a b^{n_2}) = q_2$.

The word $b^{n_1} a b^{n_2} a b^\omega$ belongs to the language. There exists an accepting state q_2 such that $\delta^*(q_0, b^{n_1} a b^{n_2} a b^{n_3}) = q_3$ **and so on**.

But there are finitely many states. Therefore there must be that some $q_i = q_j$ and hence $\delta^*(q_0, b^{n_1} a b^{n_2} \dots a b^{n_i}) = \delta^*(q_0, b^{n_1} a b^{n_2} \dots a b^{n_j})$, but this implies that there is an accepting run for the word $b^{n_1} a b^{n_2} \dots a b^{n_i} (a b^{n_{i+1}} \dots a b^{n_j})^\omega$ that contains infinitely many a's. Contradiction.☐

The need for non-determinism

Properties of the form **"eventually forever"** has exactly the shape of the ω-regular language $(a+b)^* b^\omega$.

Definition: A **persistence property** is a linear time property $P \subseteq 2^{AP}$ such that for some propositional formula φ:

$$P = \{A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \mid \exists i \geq 0 \, \forall j \geq i.\ A_j \models \varphi \}$$

A persistence property can be modeled in LTL as $\mathbf{F}\,\mathbf{G}\,\varphi$.

Alternatively, $\neg\varphi$ holds finitely many times.

Remark: $\exists i \geq 0 \, \forall j \geq i$ is sometimes written $\forall^\infty$ and can be read **"almost always"**

Generalised Büchi Automata

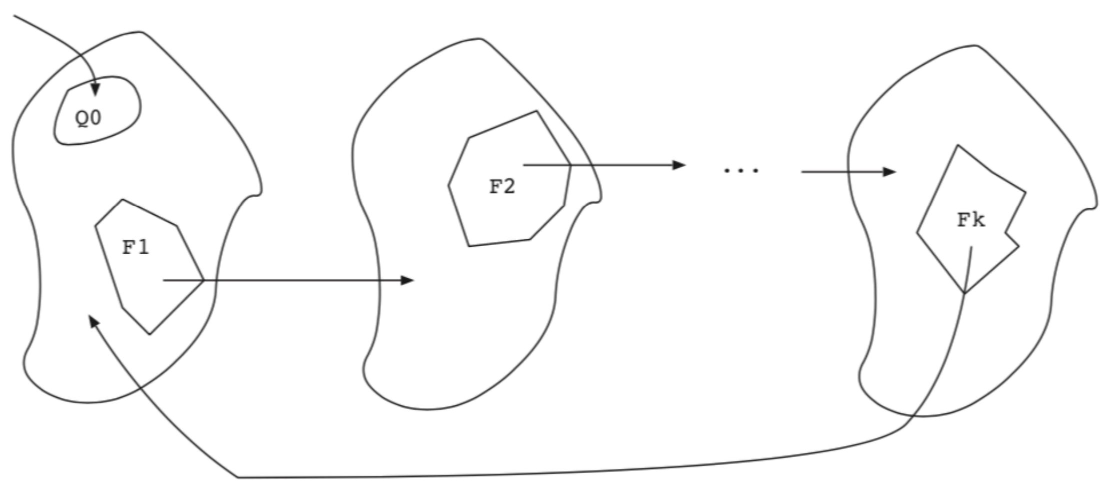
A **generelised Büchi automata** $\mathcal{A}$ is a 5-tuple $(\Sigma, Q, \delta, Q_0, \mathcal{F})$ where Σ, Q, δ, Q_0 are as for NBA, and $\mathcal{F} = \{F_1, \ldots, F_n\}$ is a possibly empty subset of 2^Q.

$F_1, \ldots, F_n$ are called *accepting sets*.

The automaton $\mathcal{A}$ **accepts** w if **there exists** an accepting run ρ over w such that for all sets $F_i \in \mathcal{F}$ we have $\inf(\rho) \cap F_i \neq \varnothing$.

Theorem. For each GNBA $\mathcal{A}$ there exists a NBA $\mathcal{A}'$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.

Proof:



Intersection of ω-regular lang.

Theorem. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are ω-regular, then $\mathcal{L}_1 \cap \mathcal{L}_2$ is ω-regular.

Proof: Given an automaton $\mathcal{A}_1 = (\Sigma, Q_1, \delta_1, I_1, F_1)$ accepting $\mathcal{L}_1$ and an automaton $\mathcal{A}_2 = (\Sigma, Q_2, \delta_2, I_2, F_2)$ accepting $\mathcal{L}_2$ we build a **generalised automata** $\mathcal{A} =(\Sigma, Q, \delta, I, \mathcal{F})$ accepting $\mathcal{L}$. We define $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_1 =(\Sigma, Q_1{\times}Q_2, \delta, I_1{\times}I_2, \{F_1{\times}Q_2, Q_1{\times}F_2\})$, where $((q_1, q_2), a, (q'_1, q'_2)) \in \delta$ iff $(q_1, a, q'_1) \in \delta_1$ and $(q_2, a, q'_2) \in \delta_2$. $\square$

We will use this trick in verification, building an **automata for the model, one for specifications** (or better, for bad behaviours) and **we will check if their intersection is empty**.

This strategy will also lead to an **alternative algorithm for LTL model checking**. There is an algorithm (based again on atoms) that allow to build an automata from an LTL formula.

Lesson 4c:

Automata Theory and Model Checking

ω-Regular Properties

Definition: A linear time property P over AP is **ω-regular** if P is an ω-regular language over the alphabet 2^{AP}.

Examples:

❖ **Invariants** are ω-regular. If φ is a property over AP defining the invariant, φ^ω is a ω-regular language.

❖ **Regular safety properties** are ω-regular.

$$(2^{AP})^\omega \setminus P_{safe} = \text{badPrefixes}(P_{safe}) \cdot (2^{AP})^\omega$$

[Remember that ω-regular are closed under complementation]

❖ Many **liveness** properties are typical examples of ω-regular (**not regular**) properties.

$((\neg crit)^* crit)^\omega$ = "a process enters critical section infinitely often"

$((\neg wait)^* wait \cdot true^* \cdot crit)^\omega + ((\neg wait)^* wait \cdot true^* \cdot crit)^* (\neg wait)^\omega$ = "whenever a process is waiting, it will enter its critical section eventually later" (**starvation freedom**)

Checking ω-regular properties

Similar to regular safety properties. However, here we have **to check language emptiness** for a (generalised) **non deterministic Büchi automata**.

Again, the idea is related to strongly connected components of a directed graph.

Definition: [**Product of a Transition System $\mathcal{M}$ and a NBA**]
Let $\mathcal{M} = (S, A, I, \rightarrow, AP, L)$ and $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be a non-blocking NBA, such that $\Sigma = 2^{AP}$. Then $\mathcal{M} \otimes \mathcal{A} = (S', A, I', \rightarrow', AP', L')$ where:

- $S' = S \times Q$
- $(s, q) \rightarrow'_a (s', q')$ whenever $s \rightarrow_a s'$ and $\delta(q, L(s'), q')$
- $I' = \{(s, q) \mid s \in I \wedge \exists q_0 \in Q_0 \,.\, (q_0, L(s), q)\}$
- $AP' = Q$
- $L': S \times Q \rightarrow 2^Q$ is given by $L'(s, q) = \{q\}$

Verifying ω-regular Properties

Let us define: $\neg\varphi = \bigwedge_{q_i \in Q} \neg q_i$ and $P_{\text{pers}} = \mathbf{F}\,\mathbf{G}\,\neg\varphi$

Theorem. Let $\mathcal{M}$ be a finite transition system and let P an ω-regular property over AP and let $\mathcal{A}$ be a nonblocking NBA such that $\mathcal{L}_\omega(\mathcal{A}) = (2^{AP})^\omega \setminus P$. Then the following are equivalent:

- $\mathcal{M} \vDash P$
- traces($\mathcal{M}$) $\cap\, \mathcal{L}_\omega(\mathcal{A}) = \varnothing$
- $\mathcal{M} \otimes \mathcal{A} \vDash P_{\text{pers}}(\mathcal{A})$

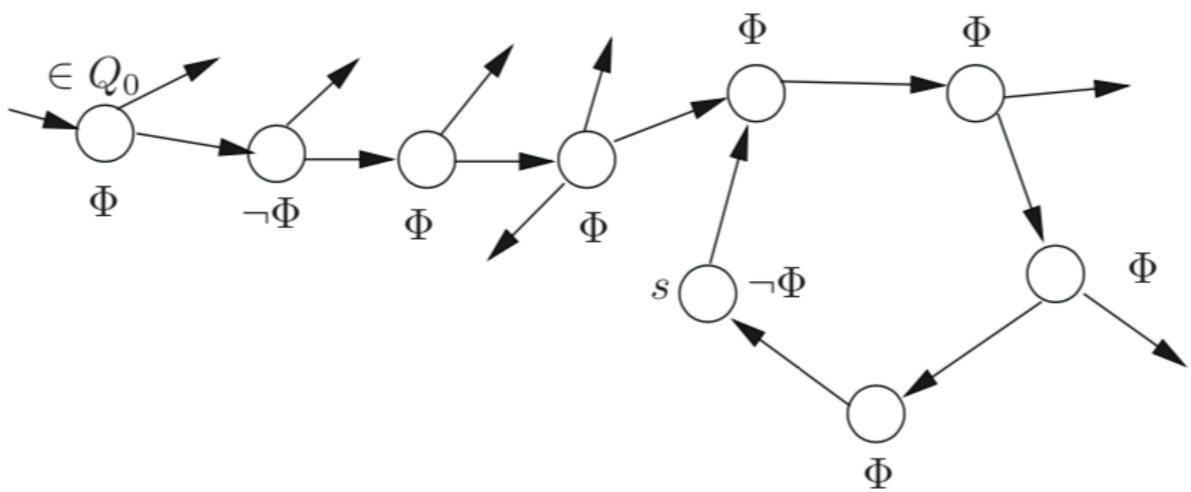
Checking a ***ω*-regular property** has been reduced to a **checking** a **persistence properties**.

Equivalently, **emptiness** of a ω-regular language is a problem of detecting **cycles:** checking whether accepting states belong to a cycle reachable from some initial state.

In this case, **counterexamples** have the form $u \cdot v^\omega$

Counterexamples

In this case, **counterexamples** have the form $u{\cdot}v^{\omega}$, where for some q in v, $L(q) \vDash \neg\varphi$.



Checking a persistence property

Once again a SCC decomposition of the graph $\mathcal{M} \otimes \mathcal{A}$ would solve the problem. traces($\mathcal{M}$) $\cap$ $\mathcal{L}_{\omega}(\mathcal{A})$= $\varnothing$ if and only if there is a SCC C that contains a state not satisfying φ and C is reachable from an initial state.

This algorithm is optimal, in the sense that it is linear with the size of $\mathcal{M} \otimes \mathcal{A}$.

However, **in practice** cycle checking can be performed more efficiently without decomposing the whole system $\mathcal{M} \otimes \mathcal{A}$ into strongly connected components.

Many model checkers implement a **nested double DFS search**. This approach has several advantages:

- When a counterexample is found, **only a small part** of $\mathcal{M} \otimes \mathcal{A}$ is visited.
- $\mathcal{M}$ is described by a program, and **states can be generated** during the nested DFS (**on-the-fly model checking**).

Double Nested DFS

```
procedure dfs1(q)
    local q';
    hash(q);
    for all successors q' of q do
        if q' not in the hash table then dfs1(q');
    if accept(q) then dfs2(q);
end procedure
```

dfs2(q) **starts when all successors of** q **have been explored**

```
procedure dfs2(q)
    local q';
    flag(q);
    for all successors q' of q do
        if q' on dfs1 stack then terminate(True);
        else if q' not flagged then dfs2(q');
        end if;
end procedure
```

State on the stack of *dfs1* up to q_1 is the finite prefix u, whereas states on the stack of *dfs2* are the cycle v.

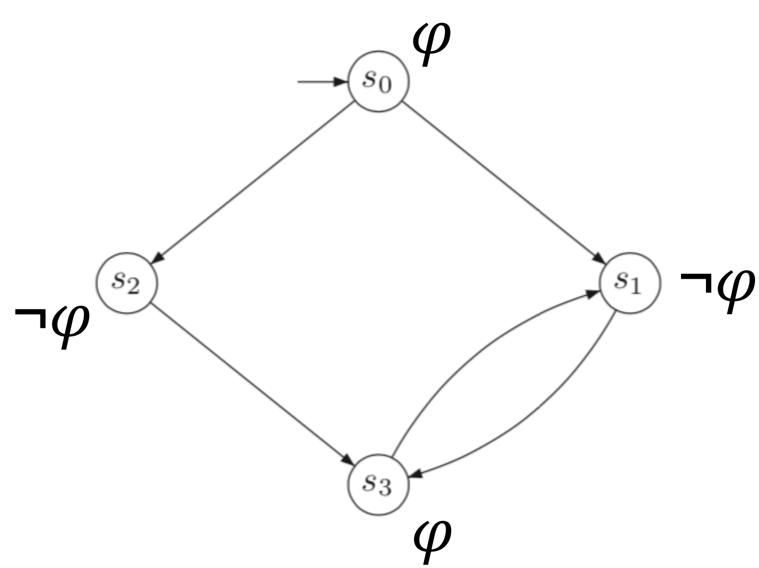
Observe that visited states by *dfs1* and *dfs2* are *global information*. This is essential to keep complexity *linear* and to avoid to visit several times the same state.

Running the Double DFS Search

Start the DFS with s_0. Consider the order of visit $s_0\ s_2\ s_3\ s_1$.

The cycle $s_1 \to s_3 \to s_1$ is found when analysing s_1. Here the inner DFS starts, because $s_1 \nvDash \varphi$ and all its successors have been already analysed. The counterexample is $s_0\ s_2\ s_3\ (s_1\ s_3\ s_1)^\omega$

The order is essential. If we start the inner DFS in s_2, we fail to find a cycle with a state already onto the stack, but we mark as visited s_3 and s_1 and therefore we later fail to find $(s_1\ s_3\ s_1)^\omega$.



Correctness of Double DFS /1

Lemma. Let q be a node that does not appear in any cycle. Then a DFS backtrack from q after all nodes reachable from q have been visited.

Theorem. The Double Nested DFS search returns a counterexample if and only if $\mathsf{traces}(\mathcal{M}) \cap \mathcal{L}_\omega(\mathcal{A}) \neq \varnothing$.

Proof: It is almost trivial to show that if the double DFS returns true, a cycle is found.

It is less obvious to show that if a cycle exists, the double DFS finds it. Or equivalently, if it returns false, no cycle exists.

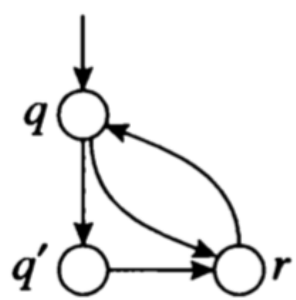
Let us suppose that there exists a cycle from q to a state on the stack of the first DFS that goes trough a state r **already flagged** by the second DFS. Let q and r the first states for which this happens and let q' be the root of the second DFS that flagged r.

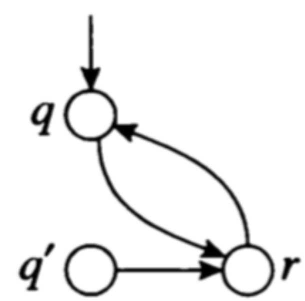
There are two cases.

Correctness of Double DFS /2

If q' is reachable from q, then there exists a cycle that would have been found examining q': $q' \to r \to q \to q'$ (see picture, left)

If q' is not reachable from q , then if q' appears on a cycle, this was missed in a previous iteration, before starting the second DFS from q, contrary to the fact that q is the first state. Thus if q' does not occur on a cycle, by the Lemma, we have discovered and backtracked from q, before starting the DFS from q'. Again, against our assumptions (see picture, right). □





Lesson 4d:

On the Fly
LTL Model Checking

LTL model checking via NBA

Theorem. For any LTL formula φ over AP, there exists a NBA $\mathcal{A}_\varphi$ wiht Words(φ)=$\mathcal{L}_\omega(\mathcal{A}_\varphi)$ which can be constructed in time and space $2^{\mathcal{O}(|\varphi|)}$.

The proof is rather technical and tedious, but the ingredients are exactly the same of the algorithm based on tableaux, see lesson **2**. In particular,

❖ automata states represent **maximal consistent sets of Cl(φ),**
❖ **transition relation** is related to presence of subformula of the form **X** ψ in Cl(φ), and
❖ **accepting states** are related to the presence of some ψ_1 **U** ψ_2 in Cl(φ). (Remember that **U** (and its negation) need to consider infinite paths).

Once one has $\mathcal{M} \otimes \mathcal{A}_{\neg\varphi}$ we just need **to check language emptiness**. ***Remark***: even though NBAs are closed under complementation, it is convenient to build $\mathcal{A}_{\neg\varphi}$ rather than complementing $\mathcal{A}_\varphi$.

On-the-fly LTL model checking

Usually, the model $\mathcal{M}$ is described by a high-level language.

The generation of reachable states of $\mathcal{M}$ can proceed in parallel with the construction of the automaton $\mathcal{A}_{\neg\varphi}$ (remembere that states of $\mathcal{M} \otimes \mathcal{A}_{\neg\varphi}$ are pairs).

The product automaton $\mathcal{M} \otimes \mathcal{A}_{\neg\varphi}$ is constructed **on demand**.

A new vertex is only considered if no accepting cycle has been found in the fragment of $\mathcal{M} \otimes \mathcal{A}_{\neg\varphi}$ already explored.

When generating the successor states in $\mathcal{A}_{\neg\varphi}$ we only need to consider those **successors matching the current state in $\mathcal{M}$**.

On-the-fly technique **is particularly effective** when a **refutation is early found**: in this case a counterexample is returned and large parts of $\mathcal{M} \otimes \mathcal{A}_{\neg\varphi}$ are not generated.

Lesson 4

That's all Folks…

…Questions?