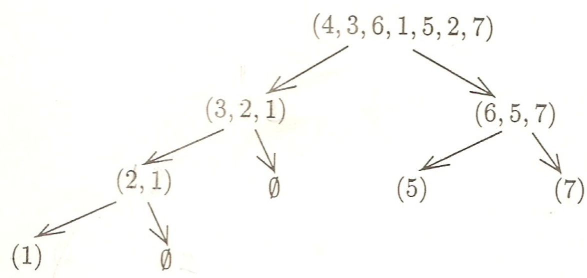
$e^{-1} > \frac{1}{3}$ for any $k \geq 1$. This leads to $|M_k| \leq 3(k+1)n$ as the theorem claims. □

Let us remark that the problem of estimating the maximum possible number of intersections of level *exactly* k is much more difficult and still unsolved. The branch of mathematics studying problems of a similar nature, i.e. combinatorial questions about geometric configurations, is called *combinatorial geometry*. A highly recommendable book for studying this subject is Pach and Agarwal [25]. A more specialized book considering problems closely related to estimating the number of intersections of level k is Sharir and Agarwal [42].

Average number of comparisons in QUICKSORT. Algorithm QUICKSORT, having received a sequence $(x_1, x_2, \ldots, x_n)$ of numbers as input, proceeds as follows. The numbers $x_2, x_3, \ldots, x_n$ are compared to x_1 and divided into two groups: those smaller than x_1 and those at least as large as x_1. In both groups, the order of numbers remains the same as in the input sequence. Each group is then sorted by a recursive invocation of the same method. The recursion terminates with trivially small groups (at most one-element ones, say). For example, the input sequence $(4, 3, 6, 1, 5, 2, 7)$ would be sorted as follows:



This algorithm may need about n^2 steps in the worst case (the worst thing we can do to the algorithm is to confront it with an already sorted sequence). In practice, however, QUICKSORT is very popular, and it is considered one of the fastest sorting algorithms. It behaves very well "on average". This is partially expressed in the following theorem.

9.4.4 Theorem. *Let $x_1 < x_2 < \cdots < x_n$ be a sequence of real numbers in an increasing order. Let π be a permutation of the set $\{1, 2, \ldots, n\}$, and let $T(\pi)$ be the number of comparisons (of pairs of elements) made by QUICKSORT for the input sequence*