SECOND PART: WIRELESS NETWORKS 2.A. FIXED NETWORKS

1

FIXED WIRELESS NETWORKS (1)

The introduction of new services, such as data communication (internet, e-mail) and video-conferencing cause shortage of capacity in existing wired networks.

A (fixed) wireless network is a collection of wireless devices (positioned *e.g.*, on buildings or towers) forming a network with radio or other wireless connections and without the aid of any established infrastructure or centralized control.

It is an alternative to the extension of the capacity of wired networks.

FIXED WIRELESS NETWORKS (2)

The advantages of fixed wireless include the ability to connect with users in remote areas without the need for laying new cables.

In rural areas, where wired infrastructure is not yet available, fixed-wireless broadband can be a viable option for Internet access.

FIXED WIRELESS NETWORKS (3)

One of the most popular applications of wireless communication is the establishment of fixed cellular telecommunication networks.

In contrast to mobile cellular networks, here the transmitters and the receivers are located at fixed points in the area of interest.

Fixed cellular networks provide a financially attractive alternative to the construction of conventional wired networks.

FIXED WIRELESS NETWORKS (4)

Fixed wireless services typically use directional radio antennas on each end of the signal.

Fixed wireless devices usually derive their electrical power from the public utility mains, unlike mobile wireless or portable wireless devices, which tend to be battery powered.

THE FREQUENCY ASSIGNMENT PROBLEM I.E. THE *L(h,k)*-LABELING OF GRAPHS

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Network Algorithms

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THE PROBLEM

THE PROBLEM (1)

Frequency assignment is necessary in many different types of wireless networks.

Depending on the particular network, the understanding of frequency assignment varies.

For this reason, several "flavors" of frequency assignment are present in the literature.

THE PROBLEM (2)

Wireless communication between two points is established with the use of a transmitter and a receiver.

The transmitter generates electrical oscillations at a radio frequency.

THE PROBLEM (3)

Issue 1.

The receiver detects these oscillations and transforms them into sounds or images, only if it is close enough to capture the signal.

In point-to-point connections, the transmitter and the receiver have to "see" each other, which means that there should be no obstacles in between them.

As a consequence, transmitters and receivers have to be built at high locations (e.g., at the roof of apartment and office buildings).

THE PROBLEM (4)

Issue 2.

The signals sent by different transmitters can interfere. Especially if signals cross each other, the use of (almost) the same frequencies should be avoided.

Reuse of frequencies may lead to a loss of quality of communication links and can be applied only for far enough transmitters.

THE PROBLEM (5)

Issue 3.

The rapid development of new wireless services (e.g., digital cellular phone networks) resulted in a run out of the most important (and expensive) resource: frequencies in the radio spectrum.

Like with all scarcely available resources, the (high) cost of frequencies implies the need for economize the use of the available frequencies.

Reuse of frequencies can offer considerable economies.

THE PROBLEM (6)

A solution to the frequency assignment problem balances the economies of reuse of frequencies and the loss of quality in the network.

Quantification of the different aspects results in a mathematical optimization problem.

THE PROBLEM (7)

In general, Frequency Assignment Problems (FAPs) have two basic aspects:

AIM: a set of wireless communication connections must be assigned some frequencies. The frequencies should be selected from a given set that may depend on the location.

CONSTRAINT: The frequencies assigned to two connections may interfere, resulting in a loss of quality of the signal.

But, what interference is?

THE PROBLEM (8)

Two conditions to have interference of two signals:

1. The two frequencies must be close on the electromagnetic band (Doppler effects) or (close to) harmonics of one another.

   The latter effect is limited, since the frequency bands from which we can choose are usually so small that they do not contain harmonics.
2. The connections must be geographically close to each other.

THE PROBLEM (9)

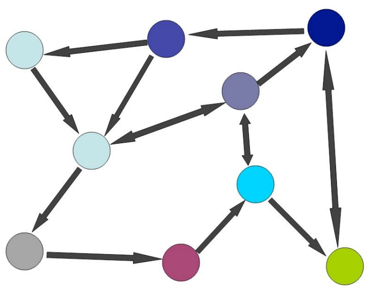
Both aspects are modeled in many different ways in the literature.

Hence: various models.

They differ in the types of constraints and in the objectives to be optimized.

Here we describe a simplified model.

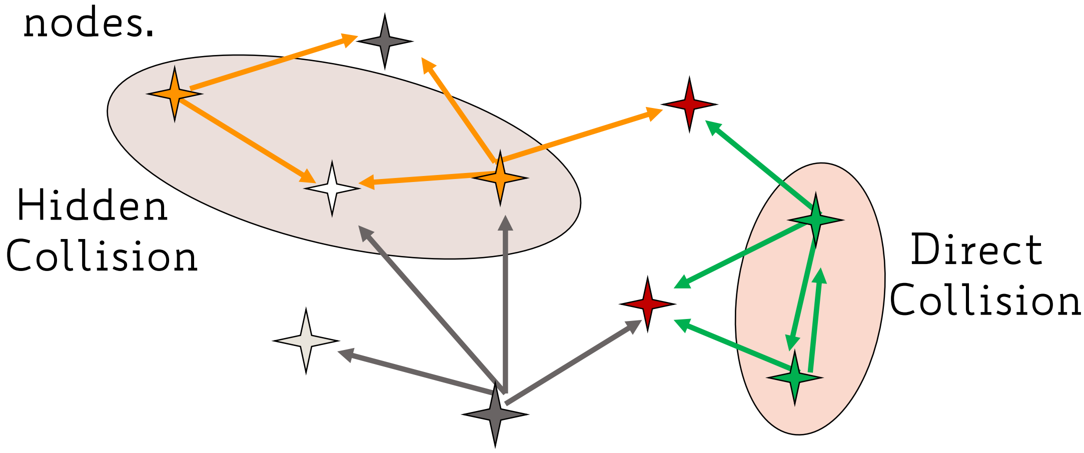
THE GRAPH MODEL



THE MODEL (1)

Interference Graph

- One node per station
- One edge between two stations if they may communicate (and hence interfere)
- Labels interpreted as channels assigned to the nodes.



THE MODEL (2)

This model consider two kinds of interference due to re-use of the same channel at "close" or "very close" sites:

Direct Collisions: stations positioned in "very close" locations receive channels at least h apart

Hidden Collisions: channels for stations positioned in "close" locations are at least k apart

↓

L(h,k)-Labeling Problem

L(*h*,*k*)-LABELING (1)

Def. Given a graph G=(V,E), an L(h,k)-labeling is a node coloring function f s.t.

- $\forall$ u, v $\in$ V, $|f(u) - f(v)| \geq h$ if (u,v) $\in$ E
- $\forall$ u, v $\in$ V, $|f(u) - f(v)| \geq k$ if $\exists$ w $\in$ V s.t. (u,w) $\in$ E and (w,v) $\in$ E

- Objective: minimizing the bandwidth $\sigma_{h,k}$
- Minimum bandwidth: $\lambda_{h,k}$

L(*h*,*k*)-LABELING (2)

Obs. The condition:

- $\forall$ u, v $\in$ V, |f(u) – f(v)|≥ k if $\exists$ w $\in$ V s.t. (u,w) $\in$ E and (w,v) $\in$ E

is often written as:

- $\forall$ u, v $\in$ V, |f(u) – f(v)| ≥ k if dist(u,v)=2

When h < k, only the first one allows a triangle to be labeled with colors at mutual distance at least max{h,k}, even if its nodes are at distance 1.
When h ≥ k the two conditions coincide.

Example: L(1,2)

L(*h*,*k*)-LABELING (3)

Usually, the minimum used color is 0.

So, an L(h,k)-labeling having span $\sigma_{h,k}$(G) uses $\sigma_{h,k}$(G)+1 different colors (slightly counter-intuitive, but it is used for historical reasons).

The problem has been introduced in the '90s with h=2 and k=1 in relation with a frequency assignment problem [Griggs e Yeh '92, Robertson '91]

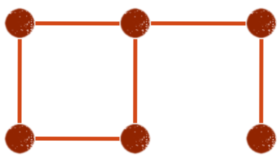
It was already known in combinatorics in the case h=1 and k=1 (coloring the square of a graph) [Wegner '77]

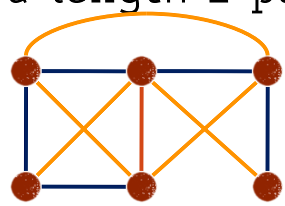
A PARENTHESIS ON THE L(*1,1*)-LABELING

When h=1 and k=1 the problem is equivalent to the classical vertex coloring (=labeling nodes s.t. adjacent nodes get different labels using min span) of the square of a graph.

Given a graph G=(V, E), its square G^2 is the graph having node set equal to V, and an edge between u and v is in G^2 iff:

- either (u,v) is in E
- or u and v are connected by a length 2 path in G





L(*h,k*)-LABELING (4)

After its definition, the L(h,k)-labeling problem has been used to model several problems:

- a kind of integer 'control code' assignment in packet radio networks to avoid hidden collisions (L(0,1)-labeling problem)
- channel assignment in optical cluster-based networks (L(0,1)- or L(1,1)-labeling depending on the fact that the clusters can contain one ore more nodes)
- more in general, channel assignment problems, with a channel defined as a frequency, a time slot, a control code, etc.

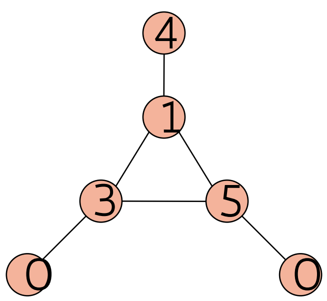
L(*h*,*k*)-LABELING (5)

The L(h,k)-labeling problem has been studied following many different approaches: graph theory and combinatorics, simulated annealing, genetic algorithms, tabu search, neural networks, …

In the following, we will survey some results.

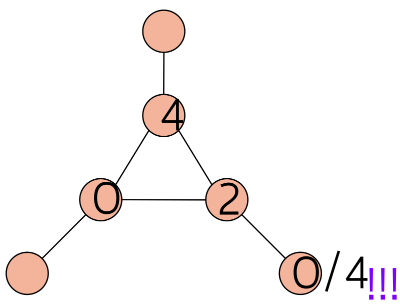
L(*h*,*k*)-LABELING (6)

Example: L(2,1)-labeling of:



$\lambda_{2,1} \leq 5$

$\lambda_{2,1}=5$: by contradiction



L(*h*,*k*)-LABELING (7)

Lemma: $\lambda_{dh,dk} = d\,\lambda_{h,k}$

Proof. Divided into 1: $\lambda_{dh,dk} \geq d\,\lambda_{h,k}$ and 2: $\lambda_{dh,dk} \leq d\,\lambda_{h,k}$.

1. $\lambda_{dh,dk} \geq d\,\lambda_{h,k}$

Let f an L(dh, dk)-labeling. Define f'=f/d.

f' is an L(h,k)-labeling and $\lambda_{dh,dk}/d = \sigma_{h,k}(f') \geq \lambda_{h,k}$.

2. $\lambda_{dh,dk} \leq d\,\lambda_{h,k}$

Similarly, let f an L(h,k)-labeling. Define f'=f d.

f' is an L(dh, dk)-labeling and $\lambda_{dh,dk} \leq \sigma_{dh,dk}(f') = d\lambda_{h,k}$.

■

So, we can restrict to use values of h and k mutually prime.

L(*h*,*k*)-LABELING (8)

PROBLEM: What if $f'=f/d$ does not produce integer values?

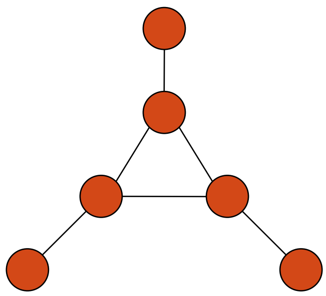
Lemma: Let x, $y \geq 0$, d>0 and k in $\mathbb{Z}^+$.

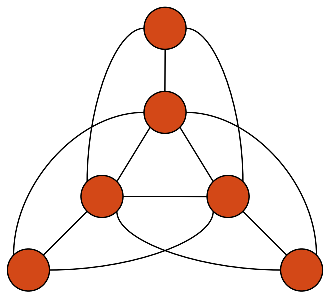
If $|x-y| \geq kd$, then $|x'-y'| \geq kd$,

where $x' = \lfloor x/d \rfloor d$ and $y' = \lfloor y/d \rfloor d$

L(*h*,*k*)-LABELING (9)

- The case k=0, for any h, is not usually considered in this context as it coincides with the classical vertex coloring.
- The case h=k is very studied in the literatue as the vertex coloring of the square of a graph.
- The case h=2k is the most studied L(h,k)-labeling problem.





NP-COMPLETENESS RESULTS (1)

The decisional version of the problem is NP-complete, even for small values of h and k:

L(0,1)-labeling of planar graphs

[Bertossi, Bonuccelli '95]

L(1,1)-labeling of general, planar, bounded degree and unit-disk graphs

[McCormick '83], [Ramanathan, Loyd '92], [Ramanathan '93], [Sen, Huson '97]

NP-COMPLETENESS RESULTS (2)

Th. The L(2,1)-labeling problem on diam. 2 graphs is NP-complete [Griggs, Yeh '92]

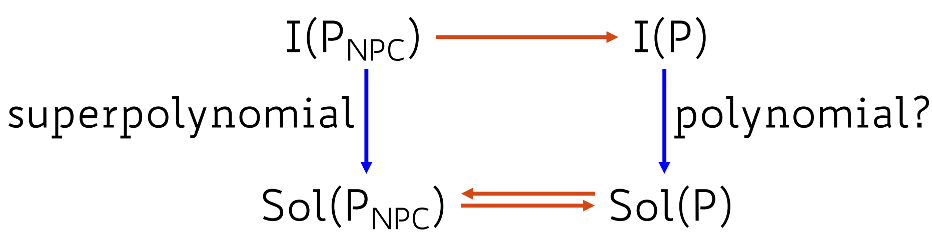
Before proving this theorem, recall that to prove the NP-completeness of a decisional problem P we have to:

- prove it is in P
- find another decisional problem P_{NPC} that is known to be NP-complete such that there exists a polynomial reduction from P_{NPC} to P

NP-COMPLETENESS RESULTS (3)

Polynomial reduction from P_{NPC} to P:
polynomial time conversion that transforms:

- any instance of P_{NPC} into an instance of P
- any "yes" solution of P_{NPC} into a "yes" solution of P
- any "no" solution of P_{NPC} into a "no" solution of P (that is: any "yes" solution of P into a "yes" solution of any "yes" solution of P_{NPC} into a "yes" solution of P)



NP-COMPLETENESS RESULTS (4)

Th. The L(2,1)-labeling problem on diam. 2 graphs is NP-complete [Griggs, Yeh '92]

Proof. Consider the following special form of the decisional problem:

DL. Instance: G=(V,E) diam. 2 graph

Question: $\lambda_{2,1}(G) \leq |V|$?

IDL. Instance: G=(V,E)

Question: Does exist an f injective s.t.

$|f(x) - f(y)| \geq 2$ if $(x,y) \in E$

and its codomain is $\{0, \ldots, |V|-1\}$?

it never maps distinct elements of its domain to the same element of its co-domain

NP-COMPLETENESS RESULTS (5)

(proof of NP-completeness cntd)

Finding a solution for IDL means finding a Hamiltonian path in G^C:

Since f is injiective, f^{-1} is defined.

Give an order to nodes:

$v_i = f^{-1}(i)$, $0 \leq i \leq |V|-1$

Observe that, since v_i and v_{i+1} cannot be adjacent in G, they are adjacent in G^C, hence

$v_0, v_1, \ldots, v_{|V|-1}$ is a Hamiltonian path.

NP-COMPLETENESS RESULTS (6)

(proof of NP-completeness cntd)

Even the reverse holds:

Given a Hamiltonian path in G^C

$v_0, v_1, \ldots, v_{|V|-1}$ define f such that $f(v_i)=i$, $0 \le i \le |V|-1$.

f is trivially injective; furthermore, given an edge $\{x,y\}$ of G, $x=v_i$ and $y=v_j$, it must hold:

$|f(x)-f(y)| \ge 2$ because x and y are not adjacent in G^C.

It follows that the two problems are equivalent.

NP-COMPLETENESS RESULTS (7)

(proof of NP-completeness cntd)

The following problem:

HP. Instance: G=(V,E)

Question: Does G have a hamiltonian path?

is NP-complete, so even IDL is NP-complete.

Instance: G=(V,E) diam. 2 graph
Question: $\lambda_{2,1}(G) \le |V|$?

DL is in NP:

We can verify in polynomial time whether a labeling f is a feasible L(2,1)-labeling for G, and whether $\lambda_{2,1}(G) \le max(f(v)) \le |V|$.

NP-COMPLETENESS RESULTS (8)

(proof of NP-completeness cntd)

Reduction from IDL to DL to prove that DL is NP-complete:

Given an instance of IDL G, construct G':

- V'=V U{x}
- E'=E U {{x,a} for each a in V}

So|V'|=|V|+1 and G' has diameter 2.

NP-COMPLETENESS RESULTS (9)

(proof of NP-completeness cntd)

We prove that from a solution for DL it is possible to deduce a solution for IDL, i.e. there is an injection f s.t.

$|f(x)- f(y)|\geq 2$ for every $(x,y) \in E$ iff $\lambda_{2,1}(G')\leq|V'|$.

- ⇒ If there exists an injection f defined on V that satisfies the condition above, define $g(v)= f(v)$ for all $v\in V$ and $g(x)=|V|+1=|V'|$.

Easily g is an L(2,1)-labeling for G' and $\lambda_{2,1}(G') \leq \max(g(v')) \leq |V'|$

NP-COMPLETENESS RESULTS (10)

(proof of NP-completeness cntd)

- ⇐ Conversely, suppose that $\lambda_{2,1}(G') \leq |V'|$, i.e. there exists a feasible L(2,1)-labeling *g* s.t. $\max(g(v')) \leq |V'| = |V|+1$.

  Observe that G′ of diam. 2 implies that $g(a) \neq g(b)$ for each $a \neq b$

  - Suppose $g(x) \neq |V|+1$ and $\neq 0$. By the property of L(2,1)-labeling, there is no v in V such that $g(v)=g(x)-1$ or $g(x)+1$. So we need $|V|+3$ labels for V′ i.e. $\lambda_{2,1}(G') \geq |V'|+1$: a contradiction.

NP-COMPLETENESS RESULTS (11)

(proof of NP-completeness cntd)

- So $g(x)$ is either 0 or $|V|+1$.
  - If $g(x)=|V|+1 \Rightarrow f(v)= g(v)$
  - If $g(x)=0 \Rightarrow f(v)= g(v)-2$

  In any case, there exists *f* injective s.t. its codomain is $\{0, \ldots, |V|-1\}$.

  The NP-compleness of DL follows. ■

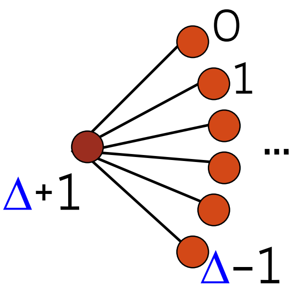
Literature in different directions:

- Lower and upper bounds for $\lambda_{h,k}$
- Limitation to special graph classes:
  - Exact labelings
  - Approximate labelings

LOWER BOUNDS (1)

- $\lambda_{2,1} \geq \Delta+1 = (\Delta-1)1+2$

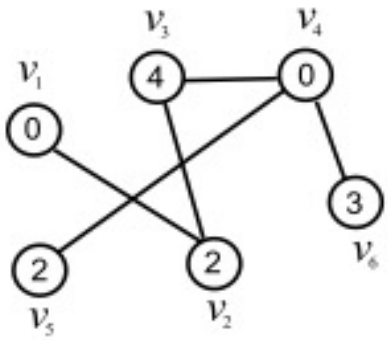
- $\lambda_{h,k} \geq (\Delta-1)k+h$ for $h \geq k$



UPPER BOUNDS (1)

Greedy Algorithm:

Given a graph G with nodes v_1, v_2, …, v_n, label its nodes in order assigning to v_i the smallest color not conflicting with the labels of its neighborhood (dist. 1 and 2)



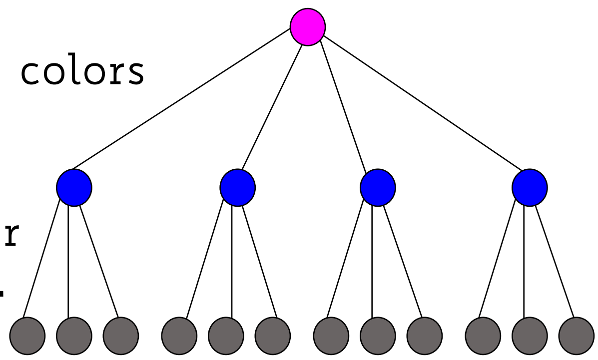
UPPER BOUNDS (2)

- Th. $\lambda_{2,1}(G) \leq \Delta^2+2\Delta$ [Griggs, Yeh '92]
- Proof.

In order to label this… →

…we eliminate at most 3 colors for each one of these… →

…and at most one color for each one of these… →

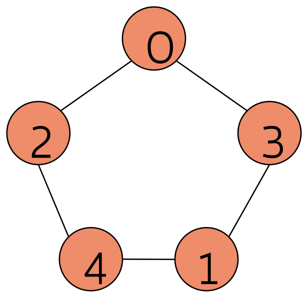


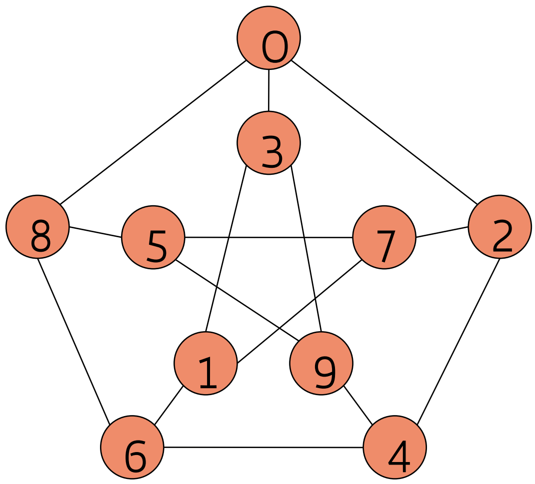
We can label all the graph with at most $1+3\Delta+(\Delta-1)\Delta$ colors.

UPPER BOUNDS (3)

Conjecture: $\lambda_{2,1}(G) \leq \Delta^2$ [Griggs, Yeh '92]

This upper bound is tight: some graphs with degree Δ, diameter 2 and Δ^2+1 nodes have λ at leats Δ^2.





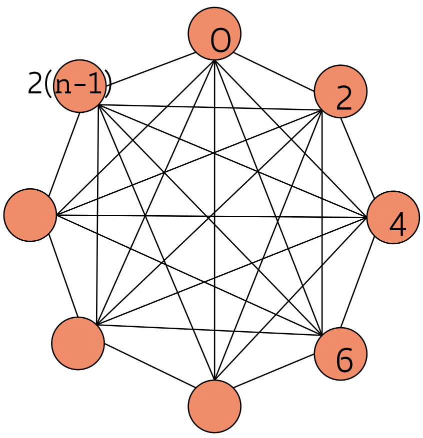
UPPER BOUNDS (4)

Conjecture: $\lambda_{2,1}(G) \leq \Delta^2$ [Griggs, Yeh '92]

- $\lambda_{2,1}(G) \leq \Delta^2+2\Delta-4$ [Jonas '93]
- $\lambda_{2,1}(G) \leq \Delta^2+\Delta$ [Chang, Kuo '96]
- $\lambda_{2,1}(G) \leq \Delta^2+\Delta-1$ [Kral, Skrekovski '03]
- $\lambda_{2,1}(G) \leq \Delta^2+\Delta-2$ [Goncalves '05]
- $\lambda_{2,1}(G) \leq \Delta^2$ for sufficiently large values of Δ [Havet, Reed and Sereni '08]

EXACT RESULTS: CLIQUES K_n

- $\lambda_{2,1}(K_n)=2(n-1)$
- All nodes are pairwise adjacent



EXACT RESULTS: STARS $K_{1,t}$

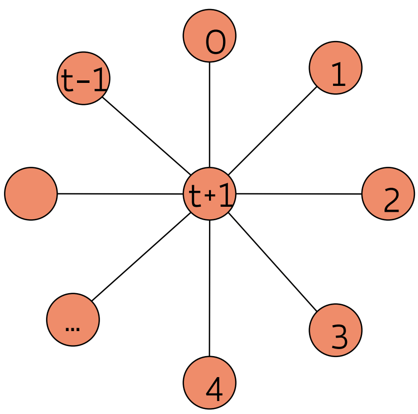
- $\lambda_{2,1}(K_{1,t})=t+1$

Proof.

- $\lambda_{2,1}(K_{1,t})\leq t+1$ easy
- $\lambda_{2,1}(K_{1,t})\geq t+1$

by contradiction:

If the center of the star is labeled with a color between 0 and t…



EXACT RESULTS: TREES T_n (1)

- $\lambda_{2,1}(T_n)=\Delta+1$ or $\Delta+2$

Proof.

- $\lambda_{2,1}(T_n) \geq \Delta+1$ because T_n contains a $K_{1,\Delta}$
- $\lambda_{2,1}(T_n) \leq \Delta+2$

  First-Fit (greedy) labeling:

  Order the nodes of T_n: $T_{n-1}=T_n- \{v_n\}$ where v_n is a leaf. In general $T_i = T_{i+1}-\{v_{i+1}\}$

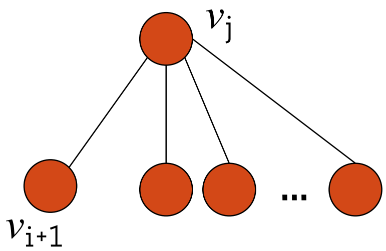
  Label v_1 with 0

  Label v_i with the first available color.

EXACT RESULTS: TREES T_n (2)

(proof: either $\lambda_{2,1}(T_n)=\Delta+1$ or $\Delta+2$ – cntd)

Assume we have already labeled all nodes from v_1 to v_i and we are going to label v_{i+1} :



- v_j parent of v_{i+1}
- necessarily $j \leq i+1$ (the nodes that are closer to the leaves have larger indices)
- v_j has at most $\Delta-1$ further adjacent nodes.

At most 3 colors are forbidden due to v_j

At most $\Delta-1$ colors are forbidden due to the nodes that are adjacent to v_j

If we have at least $(\Delta-1)+3+1$ colors, we are always able to label v_{i+1} i.e. $\lambda_{2,1}(T_n) \leq \Delta+2$. ■

EXACT RESULTS: TREES T_n (3)

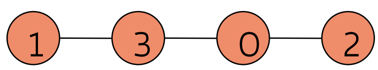
- This proof has been proposed by Griggs e Yeh ['92], who have also conjectured that it is NP-complete to decide whether the correct value is Δ+1 or Δ+2.
- Chang e Kuo ['96] have disproved this conjecture by providing a polynomial algorithm based on the dynamic programming technique and having time complexity $O(\Delta^{4.5} n)$.
- Many authors have proposed many other algorithms aiming at improving the time complexity.
- Finally, Hasunama, Ishi, Ono, and Uno ['08] proposed a linear algorithm.

EXACT RESULTS: PATHS P_n

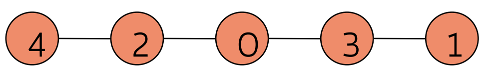
- $\lambda_{2,1}(P_2)=2$
- $\lambda_{2,1}(P_3)=3$

From the results for stars

- $\lambda_{2,1}(P_4)=3$

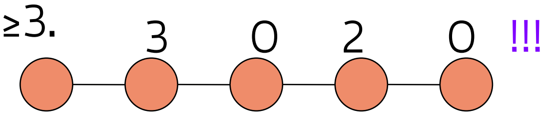


- $\lambda_{2,1}(P_n)=4$ if $n\geq5$



To prove that $\lambda_{2,1}(P_5)\leq4$:

P_5 includes a P_4 so $\lambda_{2,1}(P_5)\geq3$.
By contradiction $\lambda_{2,1}(P_5)=3$

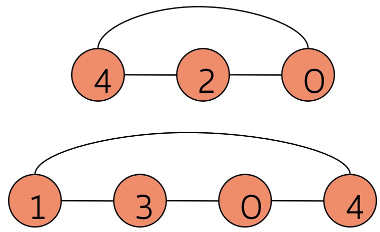


If n≥5 the result follows from the previous one and from the result for trees.

EXACT RESULTS: CYCLES C_n (1)

- $\lambda_{2,1}(C_n)=4$

If n≤4: case by case:



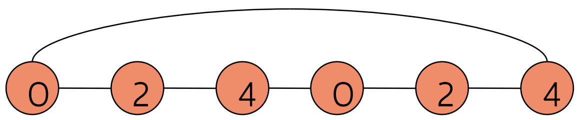
If n≥5: C_n contains P_n so $\lambda_{2,1}(C_n) \geq 4$.
It also holds $\lambda_{2,1}(C_n) \leq 4$:

3 cases: …

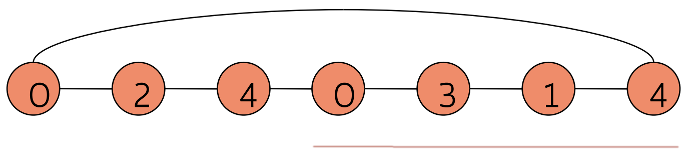
EXACT RESULTS: CYCLES C_n (2)

proof: $\lambda_{2,1}(C_n)=4$ – cntd

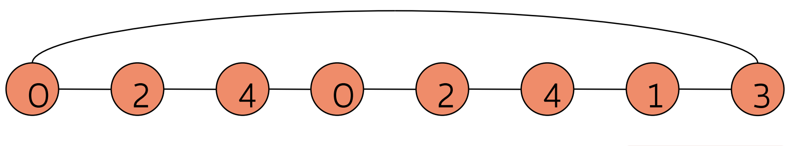
1. n=0 (mod 3)



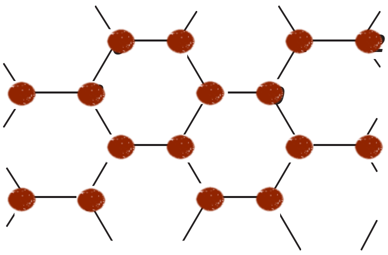
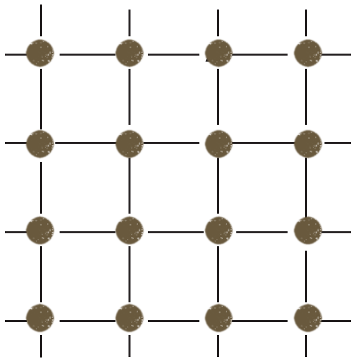
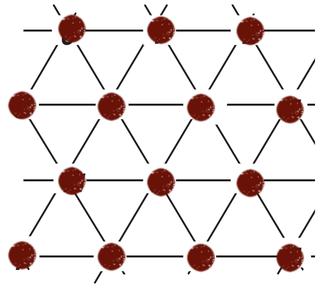
2. n=1 (mod 3)



3. n=2 (mod 3)

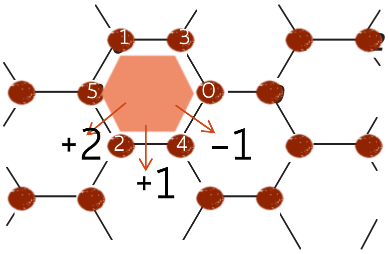


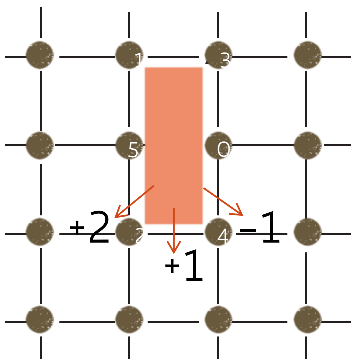
EXACT RESULTS: GRIDS (1)

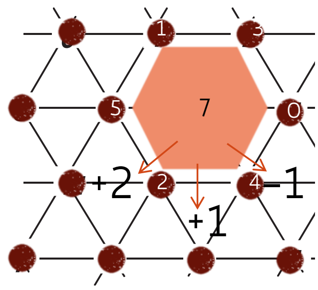




EXACT RESULTS: GRIDS (2)

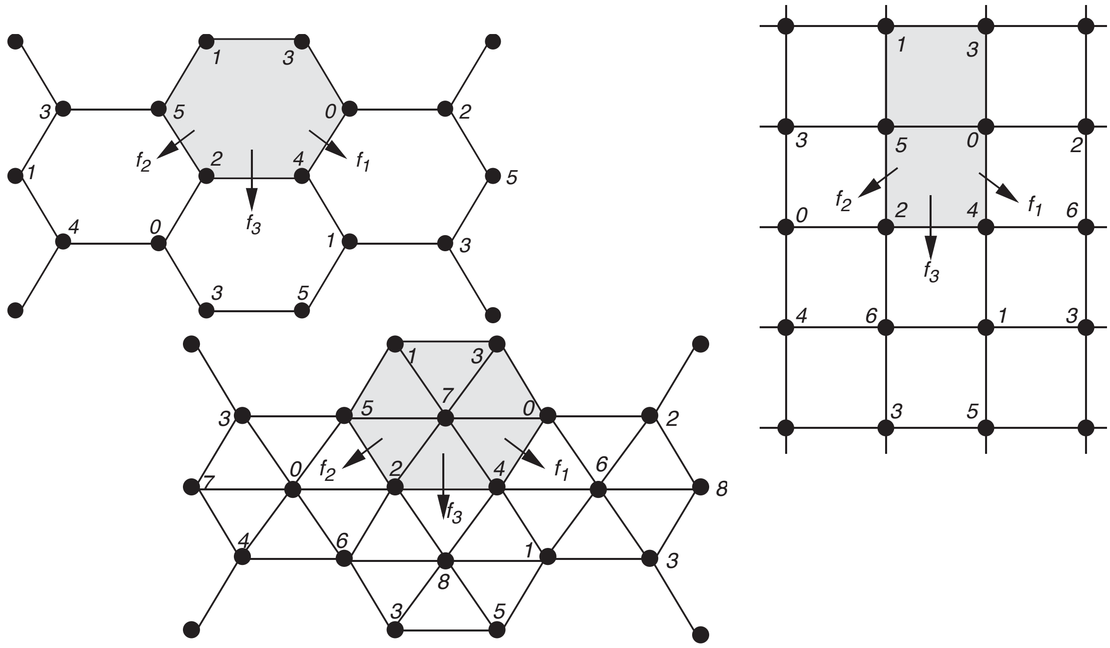
$\lambda_{2,1}(\Delta)=\Delta+2$





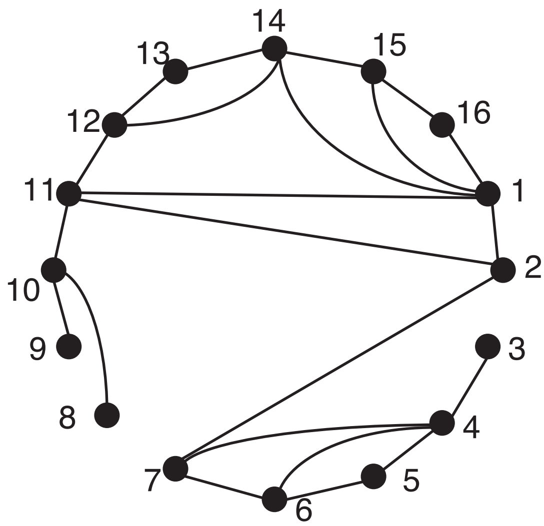


EXACT RESULTS: GRIDS (3)



APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(1)

Def. A graph is said to be outerplanar if it can be represented as a plane graph so that each node lies on the border of the external face



APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(2)

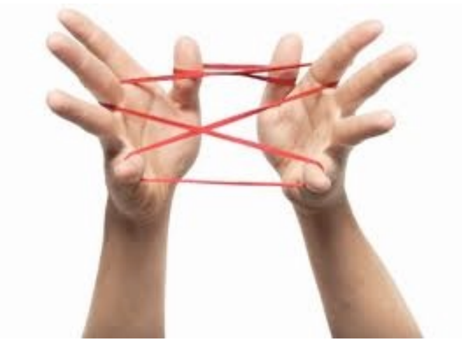
- $\lambda_{2,1}(G) \le 2\Delta+4$ because G has treewidth 2
- Jonas ['93]: $\lambda_{2,1}(G) \le 2\Delta+2$
- Bodlaender et al. ['04]: $\lambda_{2,1}(G) \le \Delta+8$ but they conjecture that $\lambda_{2,1}(G) \le \Delta+2$

☞ possible students' lesson

- C. & Petreschi ['04] $\Delta+1 \le \lambda_{2,1}(G) \le \Delta+2$ and they conjecture that this algorithm gives the optimum value

☞ possible students' lesson

VARIATIONS OF THE PROBLEM



VARIATIONS OF THE PROBLEM (1)

ORIENTED L(2,1)-LABELING

- An oriented L(2,1)-labeling of a directed graph G is a function assigning colors from 0, …,λ to the nodes of G so that nodes at distance 2 in the graph take different colors and adjacent nodes take colors at distance 2.
- Oriented L(2,1)-labeling problem
  minimizing λ
- Note. The minimum value of λ can be very different from the value of the same parameter in the undirected case. Example: trees…

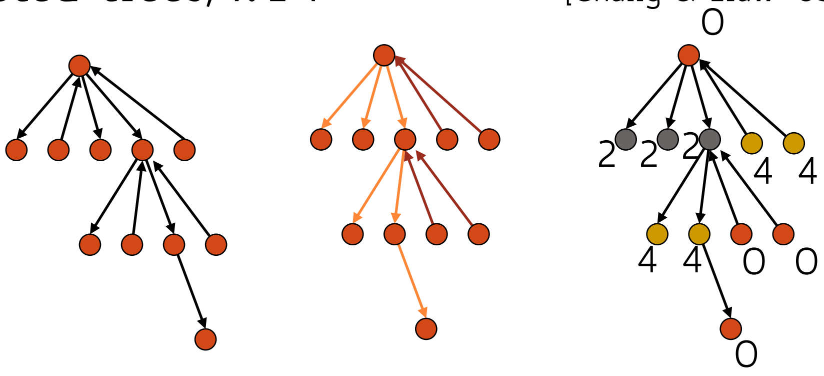
VARIATIONS OF THE PROBLEM (2)

ORIENTED L(2,1)-LABELING

- Reminder: In undirected trees, $\Delta+1 \leq \lambda \leq \Delta+2$, and the exact value is linearly decidable

  [Chang & Kuo '96, Hasunama et al. 2008]

- In directed trees, $\lambda \leq 4$ [Chang & Liaw '03]



VARIATIONS OF THE PROBLEM (3)

$L(h_1, \ldots, h_K)$-LABELING

With the aim of making the model more realistic:

- An $L(h_1, \ldots, h_k)$-labeling of G is a function assigning integer values to the nodes of the graph such that:

  $|l(u)-l(v)| \geq h_i$ if u and v are at distance I in the graph, $1 \leq i \leq k$.
- $L(h_1, \ldots, h_k)$-labeling problem: minimizing λ
- Particularly interesting: L(2,1,1) and L(δ, 1, …, 1).
- Even these special cases are NP-hard on general graphs, so special classes of graphs are handled.

VARIATIONS OF THE PROBLEM (4)

BACKBONE COLORING

If the topology has a **backbone** where the transmitting power is higher than the rest of the network:

- A Backbone coloring of a graph G wrt a graph H is a function assigning integer values to the nodes of the graph such that:

  $|l(u)-l(v)| \geq 2$ if (u,v) is an edge of H and

  $|l(u)-l(v)| \geq 1$ if (u,v) is an edge og G-H.
- Backbone coloring problem:

  minimizing λ

VARIATIONS OF THE PROBLEM (5)

n-MULTIPLE L(*h*,*k*)-LABELING

In practice, each transmitting station is able to handle more than one channel, so a set of channels is assigned to it.

Given two sets of integer values, I and J, we define:

dist(I,J)=min{|i-j|: i in I and j in J }

Example:

I={0,1,2}; J={4,5,6}; dist{I,J}=2.

VARIATIONS OF THE PROBLEM (6)

n-MULTIPLE L(*h*,*k*)-LABELING

An n-multiple L(h,k)-labeling of a graph G is a function assigning n integer values to each node of the graph so that:

dist(l(u),l(v)) ≥ h if (u,v) is an edge of G and

dist(l(u),l(v)) ≥ k if u and v are at dist. 2 in G.

n-multiple L(h,k)-labeling problem:

minimizing λ, given n.

VARIATIONS OF THE PROBLEM (7)

Some Master theses are available on variations of the vertex coloring and L(h,k)-labeling problems of special classes of graphs.

A PARENTHESIS ON THE 4 COLOR PROBLEM (1)

- Given a map, it can be naturally considered as a planar graph G.
- Given G, let G^* its dual graph:
  - Put a node of G^* in each region of G
  - Connect two nodes of G^* iff the corresponding regions (faces) are adjacent (i.e. share an edge in G)
- A vertex coloring of G^* corresponds to a map coloring of G.

A PARENTHESIS ON THE 4 COLOR PROBLEM (2)

- In fact, cartographers have always known that 4 colors were enough for each kind of map, but in 1852, Francis Guthrie wondered whether this fact could be proved.
- After more than 100 years and many (wrong) announcements, Appel and Haken proved the 4 Color Theorem in 1976.

A PARENTHESIS ON THE 4 COLOR PROBLEM (3)

- The complete proof is computer-assisted because it exhaustively examines more than 1700 configurations.
- More recently, Robertson, Sanders, Seymour, and Thomas wrote a new proof, needing to examine "only" 633 configurations.

A PARENTHESIS ON THE 4 COLOR PROBLEM (4)

There are some interesting results for other numbers of colors:

- 2-coloring.

  Polynomially solvable:

  - Assign a color to a region.
  - Assign the other color to its neighboring regions.
  - Assign the first color to its neighboring regions.
  - Continue until the regions have been all colored or there is a color conflict. In this latter case, the map is not 2-colorable.

A PARENTHESIS ON THE 4 COLOR PROBLEM (5)

- 3-coloring
  - NP-hard, hence no algorithms to decide whether a map is 3-colorable or not.
  - Method: exhaustively try all the color combinations for the regions.
  - Inapplicable: for N regions, there are 3^N possibilities. (if N=48 the combinations are about 8×10^{22})
  - ...

A PARENTHESIS ON THE 4 COLOR PROBLEM (6)

- 3-coloring (cntd)
  - There are some techniques in order to simplify the map before coloring it (for example, if a region has only 2 neighbor regions, it can be eliminated from the map: when it is re-inserted, it will be colored with the third color) but the worst case time complexity is the same.

A PARENTHESIS ON THE 4 COLOR PROBLEM (7)

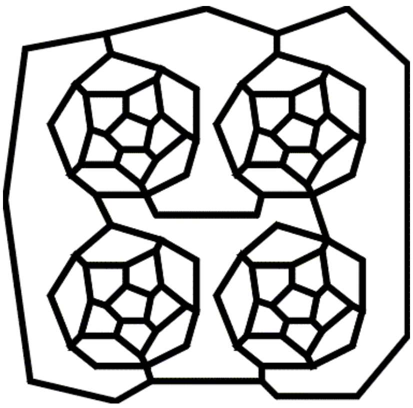
- 4-coloring
  - The proof of the 4 color theorem is constructive, and so it shows how to find a feasible coloring, but the number of cases is too high to be useful in practice.
  - There are some transformations similar to those used for the 3-coloring, but they do not eliminate the need to exhaustively try all the possibilities.

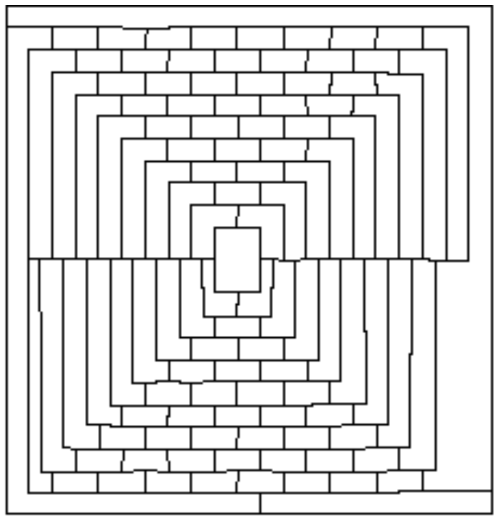
A PARENTHESIS ON THE 4 COLOR PROBLEM (8)

- 5-coloring
  - It is relatively easy to color a map using 5 colors. There is an algorithm that first simplifies the map, eliminating all the regions and then re-inserts them, assigning the correct color.

A PARENTHESIS ON THE 4 COLOR PROBLEM (9)

We conclude with a puzzle:
Try to 4-color these 2 maps...





In 1975 Martin Gardner claimed he could prove that this map was not 4-colorable (April fool)

A PARENTHESIS ON THE 4 COLOR PROBLEM (9)

Solutions