## SECOND PART: <br> Wireless Networis 2.I. AD HOC NETWORKS

## THE FREQUENCY ASSIGNMENT PROBLEM I.E. <br> THE $L(h, k)$ LABELLNG OF GRAPHS

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## THE PROBLEM (1)



- There is not "the" frequency assignment problem.
- Frequency assignment is necessary in many different types of wireless networks.
- Depending on the particular network, the understanding of frequency assignment varies.
- For this reason, several "flavors" of frequency assignment are present in the literature.


## THE PROBLEM (2)

- Wireless communication between two points is established with the use of a transmitter and a receiver.
- The transmitter generates electrical oscillations at a radio frequency.
- The receiver detects these oscillations and transforms them into sounds or images, IF it is close enough to capture the signal.
- When two transmitters use the same frequency, they may interfere.


## THE PROBLEM (4)

The introduction of new services, such as data communication (internet, e-mail) and videoconferencing cause shortage of capacity in existing wired networks.
Point-to-point wireless connections can be used as an alternative to the extension of the capacity of these wired networks.

In both cases no cable connections have to be established.

## THE PROBLEM (3)

One of the most popular applications of wireless communication is the establishment of fixed cellular telecommunication networks.
In contrast to mobile cellular networks, in non-mobile or fixed systems both the transmitters and the receivers are located at fixed points in the area of interest.
Fixed cellular networks provide a financially attractive alternative to the construction of conventional wired networks.

## THE PROBLEM (5)

In point-to-point connections, the transmitter and the receiver have to "see" each other, which means that there should be no obstacles in between them.

As a consequence, transmitters and receivers have to be built at high locations (e.g., at the roof of apartment and office buildings).

Nevertheless, the signals sent by different transmitters can interfere. Especially if signals cross each other, the use of (almost) the same frequencies should be avoided.

## THE PROBLEM (6)

The rapid development of new wireless services (e.g. digital cellular phone networks) resulted in a run out of the most important (and expensive) resource: frequencies in the radio spectrum.

Like with all scarcely available resources, the cost of frequency-use provides the need for economic-use of the available frequencies.

Reuse of frequencies within a wireless communication network can offer considerable economies.

## THE PROBLEM (8)

In general, Frequency Assignment Problems (FAPs) have two basic aspects:

1. a set of wireless communication connections must be assigned some frequencies such that, for every connection, data transmission between the transmitter and the receiver is possible. The frequencies should be selected from a given set that may depend on the location.
2. The frequencies assigned to two connections may interfere, resulting in a loss of quality of the signal.

But, what interference is?

## THE PROBLEM (7)

However, reuse of frequencies may also lead to a loss of quality of communication links.

Namely, the use of (almost) the same frequency for multiple wireless connections can cause an interference between the signals that is unacceptable.
A solution to the frequency assignment problem balances the economies of reuse of frequencies and the loss of quality in the network.

Quantification of the different aspects results in a mathematical optimization problem.

## THE PROBLEM (9)

Two conditions must be fulfilled in order to have interference of two signals:
a) The two frequencies must be close on the electromagnetic band (Doppler effects) or (close to) harmonics of one another.
The latter effect is limited, since the frequency bands from which we can choose are usually so small that they do not contain harmonics.
b) The connections must be geographically close to each other.

## THE PROBLEM (10)

Both aspects are modeled in many different ways in the literature.

Hence: various models.
They differ in the types of constraints and in the objectives to be optimized.
Here we describe a simplified model.

## THE GRAPH MODEL



## THE MODEL (2)

This model consider two kinds of interference due to re-use of the same channel at "close" or "very close" sites:

Direct Collisions: stations positioned in "close" locations receive channels at least $h$ apart
Hidden Collisions: channels for stations positioned in "very close" locations are at least $k$ apart

## $\mathrm{L}(h, \mathrm{~A}$ - $-\mathrm{LABELLING}(1)$

## Interference Graph

- One node per station
- One edge between two stations if they may communicate (and hence interfere) $\rightarrow$ very close locations correspond to adjacent nodes)
- Labels interpreted as channels assigned to the nodes.

```
f: node coloring function s.t.
- }\forallu,v\inV|f(u)-\boldsymbol{f}(v)|\geqh\mathrm{ if (u,v) &E
- \forallu,v\inV |f(u)-f(v)| \k if }\existsw\inV s.t. (u,w) \inE
and (w,v) \inE
    Objective: minimizing the bandwidth }\mp@subsup{\sigma}{h,k}{
        Minimum bandwidth: }\mp@subsup{\lambda}{h,k}{
```


## $\mathrm{L}(h, \mathrm{~A}$ - $-\mathrm{LABBELLNG}(3)$

Usually, the minimum used color is 0 .
So, an $L(h, k)$-labeling having span $\sigma_{h, k}(G)$ uses $\sigma_{h, k}(G)+1$ different colors.
This is slightly counter-intuitive, but it is used for historical reasons.
The problem has been introduced in the '90s with $h=2$ and $k=1$ in relation with a frequency assignment problem
[Griggs e Yeh '92, Robertson '91]
This problem was already known in combinatorics in the case $h=1$ and $k=1$ (coloring the square of a graph)

## $\mathrm{L}(h, A)$-LABELING (2)

Obs. The condition:
$-\forall u, v \in V|\boldsymbol{f}(u)-\boldsymbol{f}(v)| \geq \boldsymbol{k}$ if $\exists w \in V$ s.t. $(u, w) \in E$ and $(w, v) \in E$
is often written as:

$$
-\forall u, v \in V|\boldsymbol{f}(u)-\boldsymbol{f}(v)| \geq k \text { if } \operatorname{dist}(u, v)=2
$$

The first one works both when $h \geq k$ and when $h<k$.
It allows a triangle to be labeled with colors at mutual distance at least $\max \{h, k\}$, even if its nodes are at distance 1. When $h \geq k$ the two conditions coincide.

Example: $L(1,2)$


## A PARENTHESIS ON THE L( 1,1 )-LABBELING (1)

When $h=1$ and $k=1$ the problem is equivalent to the classical vertex coloring (=labeling nodes s.t. adjacent nodes get different labels using min span) of the square of a graph.

Given a graph $G=(V, E)$, its square $G^{2}$ is defined as a graph having node set equal to $V$, and an edge between $u$ and $v$ is in $G^{2}$ iff:
either $(u, v)$ is in $E$ or $u$ and $v$ are connected by a length 2 path in $G$


## A PARENTHESIS ON THE L(I, I)-LABELING (2)

If the graph is stored in an adjacency matrix:


010100
101010
$A(G)=010001$
010100
001000

Compute $A(G)^{2}$ : $\left(a_{i j}\right)^{2}=\mathrm{S}_{k=1 \ldots n} a_{i k} a_{k j}$

201020 030201
$A(G)^{2}=102010$ 201020 010001
$\left(a_{i j}\right)^{2}=x$ iff there are $x$ 2-length paths between $i$ and $j$. To store together the knowledge about 1-and 2-length paths: $A(G)^{2}+A(G)->A\left(G^{2}\right)$

## $\mathrm{L}(h, k)$-LABELLNG ( 5 )

The $L(h, k)$-labeling problem has been studied following many different approaches: graph theory and combinatorics, simulated annealing, genetic algorithms, tabu search, neural networks, ...

In the following, we will survey some results.

## $\mathrm{L}(h, k)$-LABELLING (4)

After its definition, the $L(h, k)$-labeling problem has been used to model several problems:

- a kind of integer 'control code' assignment in packet radio networks to avoid hidden collisions ( $L(0,1$ )labeling problem)
- channel assignment in optical cluster-based networks ( $L(0,1)$ - or $L(1,1)$-labeling depending on the fact that the clusters can contain one ore more nodes)
- more in general, channel assignment problems, with a channel defined as a frequency, a time slot, a control code, etc.


## $\mathrm{L}(h, h)-\mathrm{LABELING}(6)$

Example: $L(2,1)$-labeling of:

$\lambda_{2,1} \leq 5$
$\lambda_{2,1}=5$ : by contradiction

!!!

## $\mathrm{L}(h, k)$-LABELING (7)

Lemma: $\lambda_{\text {dh }, d k}=d \lambda_{h, k}$
Proof. Divided into two parts: $\lambda_{d h, d k} \geq d \lambda_{h, k}$ and $\lambda_{d h, d k} \leqslant d \lambda_{h, k}$.

1. $\lambda_{d h, d k} \geq d \lambda_{h, k}$

Let $f$ an $L(d h, d k)$-labeling. Define $f^{\prime}=f / d$.
$f^{\prime}$ is an $L(h, k)$-labeling and $\lambda_{d h, d k} / d=\sigma_{h, k}\left(f^{\prime}\right) \geq \lambda_{h, k}$.
2. $\lambda_{d h, d k} \leqslant d \lambda_{h, k}$

Similarly, let $f$ an $L(h, k)$-labeling. Define $f^{\prime}=f d$.
$f^{\prime}$ is an $L(d h, d k)$-labeling and $\quad \lambda_{d h, d k} \leq \sigma_{d h, d k}\left(f^{\prime}\right)=d \lambda_{n, k}$.

It follows that we can restrict to use values of $h$ and $k$ mutually prime.

## $\mathrm{L}(h, k)$-LABELLNG ( 9 )

- The case $k=0$, for any $h$, is not usually considered as an $L(h, k)$-labeling problem, as it coincides with the classical vertex coloring.
- The case $h=k$ is very studied in the literatue as the vertex coloring of the square of a graph.
- The case $h=2 k$ is the most studied $L(h, k)$-labeling problem.



## $\mathrm{L}(h, k)$-LABELING ( 8 )

PROBLEM: What if $f^{\prime}=f / d$ does not produce integer values?

Lemma: Let $x, y \geq 0, d>0$ and $k$ in $\mathbb{Z}^{+}$.
If $|x-y| \geq k d$, then $\left|x^{\prime}-y^{\prime}\right| \geq k d$,
where $x^{\prime}=\lfloor x / d\rfloor d$ and $y^{\prime}=\lfloor y / d\rfloor d$
It follows we can restrict to use values of $h$ and $k$ integer and mutually prime.

## NP-COMPLETENESS RESULTS (1)

The decisional version of the problem is NP-complete, even for small values of $h$ and $k$ :
$L(0,1)$-labeling of planar graphs
[Bertossi, Bonuccelli '95]
$L(1,1)$-labeling of general, planar, bounded degree and e unit-disk graphs
[McCormick '83], [Ramanathan, Loyd '92],
[Ramanathan ‘93], [Sen, Huson '97]

## NP-COMPLETENESS RESULTS (2)

Th. The $L(2,1)$-labeling problem on diam. 2 graphs is NPcomplete [Griggs, Yeh '92]
Proof. Consider the following special form of the decisional problem:
it never maps distinct elements of its
domain to the same element of its co-domain
IDL. Instance: $G=(V, E)$
Question: Does exist an $f$ injective s.t.

$$
|f(x)-f(y)| \geq 2 \text { if }(x, y) \in E
$$

and its codomain is $\{0, \ldots,|V|-1\}$ ?

## NP-COMPLETENESS RESULTS (4)

(proof of NP-completeness cntd)

Even the reverse holds:
Given a Hamiltonian path in $G^{C}$
$v_{0}, v_{1}, \ldots, v_{\mid V-1}$ define $f$ such that $f\left(v_{i}\right)=i, 0 \leq i \leq|V|-1$.
$f$ is trivially injective; furthermore, given an edge $\{x, y\}$ of $G$, $x=v_{i}$ and $y=v_{j}$, it must hold:
$|f(x)-f(y)| \geq 2$ because $x$ and $y$ are not adjacent in $G^{C}$.
It follows that the two problems are equivalent.

## NP-COMPLETENESS RESULTS (3)

(proof of NP-completeness cntd)

Finding a solution for IDL means finding a Hamiltonian path in $G^{c}$ :

Since $f$ is injiective, $f^{-1}$ is defined.
Give an order to nodes:
$v_{i}=f^{-1}(i), 0 \leq i \leq|V|-1$
Observe that, since $v_{i}$ and $v_{i+1}$ cannot be adjacent in $G$, they are adjacent in $G^{C}$, hence
$v_{0}, v_{1}, \ldots, v_{\backslash \backslash-1}$ is a Hamiltonian path.

## NP-COMPLETENESS RESULTS (5)

(proof of NP-completeness cntd)

The following problem:
HP. Instance: $G=(V, E)$
Question: Does $G$ have a hamiltonian path?
is NP-complete, so even IDL is NP-complete.
Instance: $G=(V, E)$ diam. 2 graph
DL is in NP:
Question: $\lambda_{2,1}(G) \leq \mid V$ ?
We can verify in polynomial time that $G$ has diameter 2, whether a labeling $f$ is a feasible $L(2,1)$-labeling, and whether $\lambda_{2, l}(G) \leq\|f(G)\| \leq \mid V /$.

## NP-COMPLETENESS RESULTS (6)

(proof of NP-completeness cntd)

Transformation from IDL to $D L$ to prove that $D L$ is
NP-complete:
Given an instance of IDL G, construct G':

- $V^{\prime}=V U\{x\}$
- $E^{\prime}=E \cup\{\{x, a\}$ for each $a$ in $V\}$

So| $V^{\prime}\left|=|V|+1\right.$ and $G^{\prime}$ has diameter 2

## NP-COMPLETENESS RESULTS (8)

(proof of NP-completeness cntd)

- $\Leftarrow$ Conversely, suppose that $\lambda_{2,1}\left(G^{\prime}\right) \leq\left|V^{\prime}\right|$, i.e. there exists a feasible $L(2,1)$-labeling $g$ s.t. $\| g\left(V^{\prime}\right)| | \leq\left|V^{\prime}\right|=|V|+1$.
Observe that $G$ ' of diam. 2 implies that $g(a) \neq g(b)$ for each $a \neq b$
- Suppose $g(x) \neq|V|+1$ and $\neq 0$. By the property of $L(2,1)-$ labeling, there is no $v$ in $V$ such that $g(v)=g(x)-1$ or $g(x)+1$. So we need $|V|+3$ labels for $V^{\prime}$ i.e. $\lambda_{2,1}\left(G^{\prime}\right) \geqslant\left|V^{\prime}\right|+1$ : a contradiction.


## NP-COMPLETENESS RESULTS (7)

(proof of NP-completeness cntd)

We prove that from a solution for DL it is possible to deduce a solution for IDL, i.e. there is an injection $f$ s.t.
$|f(x)-f(y)| \geq 2$ for every $(x, y) \in E$ iff $\lambda_{2,1}\left(G^{\prime}\right) \leq\left|V^{\prime}\right|$.

- $\Rightarrow$ If there exists an injection $f$ defined on $V$ that satisfies the condition above, define $g(v)=f(v)$ for all $v \in V$ and $g(x)=|V|+1=\left|V^{\prime}\right|$.
Easily $g$ is an $L(2,1)$-labeling for $G^{\prime}$ and

$$
\lambda_{2,1}\left(G^{\prime}\right) \leq\left\|g\left(G^{\prime}\right)\right\| \leq\left|V^{\prime}\right|
$$

## NP-COMPLETENESS RESULTS (9)

(proof of NP-completeness cntd)

- So $g(x)$ is either 0 or $|V|+1$.
- If $g(x)=|V|+1=>f(v)=g(v) O K$
- If $g(x)=0=>f(v)=g(v)-2$ OK

In any case, there exists $f$ injective s.t. its codomain is $\{0, \ldots,|V|-1\}$.

The NP-compleness of DL follows.

## Literature in different directions:

- Lower and upper bounds for $\lambda_{h, k}$
- Limitation to special graph classes:
- Exact labelings
- Approximate labelings


## LOWER BOUNDS (2)

- $\exists G$ s.t. $\lambda_{2,1}(G) \geq \Delta^{2}-\Delta$
[Griggs, Yeh ‘92]
Incidence graph of a projective plane $\pi(n)$ of order $n$,
$G=(U \cup V, E)$ s.t.
- $U\left|=|V|=n^{2}+n+1\right.$
$u \in U$ corresonds to a point $P_{u}$ of $\pi(n)$
- $v \in V$ corresonds to a line $I_{v}$ of $\pi(n)$
$E=\left\{(u, v)\right.$ s.t. $\left.P_{u} \in I_{v}\right\}$
- $\lambda_{2,1} \geq \Delta+1=(\Delta-1) 1+2$
- $\lambda_{h, k} \geq(\Delta-1) k+h$ for $h \geq k$



## LOWER BOUNDS (3)

- $G$ is regular and $\Delta=n+1$
- For each pair of nodes in $U$ (or in $V$ ), their distance is 2 ,
- $\forall u, v \in U(\in V),|\operatorname{Adj}(u) \cap \operatorname{Adj}(v)|=1$
$\Rightarrow \lambda_{2,1}(G) \geq|U|-1=|V|-1=\Delta^{2}-\Delta$


## UPPER BOUNDS (1)

## Greedy Algorithm:

Given a graph $G$ with nodes $v_{1}, v_{2}, \ldots, v_{n}$,
label its nodes in order assigning to $v_{i}$ the smallest color not conflicting with the labels of its neighborhood (dist. 1 and 2)


## UPPER BOUNDS (3)

Conjecture: $\lambda_{2,1}(G) \leq \Delta^{2}$
[Griggs, Yeh ‘92]

This upper bound is tight: some graphs with degree $\Delta$, diameter 2 and $\Delta^{2}+1$ nodes have $\lambda$ at leats $\Delta^{2}$.

(B)

- Th. $\lambda_{2,1}(G) \leq \Delta^{2}+2 \Delta$
[Griggs, Yeh ‘92]
- Proof.

In order to label this
...we eliminate at most 3 colors for each one of these...
...and at most one color for each one of these...

We can label all the graph with at most $1+3 \Delta+(\Delta-1) \Delta$ colors.

## UPPER BOUNDS (4)

Conjecture: $\lambda_{2,1}(G) \leq \Delta^{2}$
[Griggs, Yeh '92]

- $\lambda_{2,1}(G) \leq \Delta^{2}+2 \Delta-4$ [Jonas '93]
- $\lambda_{2,1}(G) \leq \Delta^{2}+\Delta$
[Chang, Kuo '96]
- $\lambda_{2,1}(\mathrm{G}) \leq \Delta^{2}+\Delta-1$
[Kral, Skrekovski ‘03]
- $\lambda_{2,1}(G) \leq \Delta^{2}+\Delta-2$
[Goncalves '05]
- $\lambda_{2,1}(G) \leq \Delta^{2}$ for sufficiently large values of $\Delta$ [Havet, Reed and Sereni '08]


## EXACT RESULTS: CLIQUES $\Pi_{n}$

- $\lambda_{2,1}\left(K_{n}\right)=2(n-1)$
- All nodes are pairwise adjacent



## EXACT RESULTS: TREES $T_{n}(1)$

- $\lambda_{2,1}\left(T_{n}\right)=\Delta+1$ or $\Delta+2$

Proof.

- $\lambda_{2,1}\left(T_{n}\right) \geq \Delta+1$ because $T_{n}$ contains a $K_{1, \Delta}$
- $\lambda_{2,1}\left(T_{n}\right) \leq \Delta+2$
first-fit (greedy) labeling:
Order the nodes of $T_{n}: T_{n-1}=T_{n}-\left\{v_{n}\right\}$ where $v_{n}$ is a leaf. In general $T_{i}=T_{i+1}-\left\{v_{i+1}\right\}$
Label $v_{1}$ with 0 .
Label $v_{i}$ with the first available color.


## EXACT RESULTS: STARS $K_{L, t}$

- $\lambda_{2,1}\left(K_{1, t}\right)=t+1$

Proof.

- $\lambda_{2,1}\left(K_{1, t}\right) \leq t+1$ easy
- $\lambda_{2,1}\left(K_{1, t}\right) \geq t+1$
by contradiction:
If the center of the star is labeled with a color different from 0 and from $t+1 \ldots$



## EXACT RESULTS: TREES $T_{n}(2)$ <br> (proof: either $\lambda_{2,1}\left(T_{n}\right)=\Delta+1$ or $\Delta+2-$ cntd)

Assume we have already labeled all nodes from $v_{1}$ to $v_{i}$ and we are going to label $v_{i+1}$ :

$v_{j}$ parent of $v_{i+1}$
necessarily $j \leq i+1$ (the nodes that are closer to the leaves have larger indices)
$v_{j}$ has at most $\Delta-1$ further adjacent nodes.

At most 3 colors are forbidden due to $v_{j}$ At most $\Delta-1$ colors are forbidden due to the nodes that are adjacent to $v_{j}$
If we have at least $(\Delta-1)+3+1$ colors, we are always able to label $v_{i+1}$ i.e. $\lambda_{2,1}\left(T_{n}\right) \leq \Delta+2$.

## EXACT RESULTS: TREES $T_{n}(3)$

- This proof has been proposed by Griggs e Yeh ['92], who have also conjectured that it is NP-complete to decide whether the correct value is $\Delta+1$ or $\Delta+2$.
- Chang e Kuo ['96] have disproved this conjecture by providing a polynomial algorithm based on the dynamic programming technique and having time complexity $O\left(\Delta^{4.5} n\right)$.
- Many authors have proposed many other algorithms aiming at improving the time complexity.
- Finally, Hasunama, Ishi, Ono, Uno ['08] have proposed a linear algorithm.


## EXACT RESULTS: CYCLES $C_{n}^{\prime}(1)$

- $\lambda_{2,1}\left(C_{n}\right)=4$

If $n \leq 4$ : case by case:


If $n \geq 5$ : $C_{n}$ contains $P_{n}$ so $\lambda_{2,1}\left(C_{n}\right) \geq 4$.
It also holds $\lambda_{2,1}\left(C_{n}\right) \leq 4$ :
3 cases: ...

## EXACT RESULTS: PATHS $P_{n}$

- $\lambda_{2,1}\left(P_{2}\right)=2$
- $\lambda_{2,1}\left(P_{3}\right)=3$From the results for stars
- $\lambda_{2,1}\left(P_{4}\right)=3$

- $\lambda_{2,1}\left(P_{n}\right)=4$ if $n \geq 5$

$P_{5}$ includes a $P_{4}$ so $\lambda_{2,1}\left(P_{5}\right) \geq 3$. By contradiction $\lambda_{2,1}\left(P_{5}\right)=3$


If $n \geq 5$ the result follows from the previous one and from the result for trees.

## EXACT RESUITS: CYCLES $\mathcal{C}_{n}(2)$

(proof: $\lambda_{2,1}\left(C_{n}\right)=4-$ cntd)

1. $n=0(\bmod 3)$

2. $n=1(\bmod 3)$

3. $n=2(\bmod 3)$


## EXACT RESULTS: GRIDS (1)


(3)

## EXACT RESULTS: GRIDS (3)


©

## EXACT RESULTS: GRIDS (2)

$$
\lambda_{2, l}(\Delta)=\Delta+2
$$


(54)

## APPROXIMATL RESUITS: OUTERPLANAR GRAPHS(1)

- $\lambda_{2,1}(G) \leq 2 \Delta+4$ because $G$ has treewidth 2
- Jonas ['93]: $\lambda_{2,1}(G) \leq 2 \Delta+2$
- Bodlaender et al. ['04]: $\lambda_{2,1}(G) \leq \Delta+8$ but they conjecture that $\lambda_{2,1}(G) \leq \Delta+2$-> possible students' lesson
- C.\& Petreschi ['04] $\Delta+1 \leq \lambda_{2,1}(G) \leq \Delta+2$ and they conjecture that this algorithm gives the optimum value.


## APPROXIMATL RESULTS: OUTERPLANAR GRAPHS(2)

Def. A graph is said to be outerplanar if it can be represented as a plane graph so that each node lies on the border of the external face

(2)

## APPROXIMATE RESUITS: OUTERPLANAR GRAPHS(4)



## APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(3)

- Ordered Breadth First Tree:
- Choose a node $r$
- Induce a total order on the nodes on the external face
- Run a BFS from $r$ so that nodes coming before in the ordering are visited before than the others.

(3)


## APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(5)

Th. (well known)
$\mathrm{G}=(\mathrm{V}, \mathrm{E}) ; B F T T=\left(V, E^{\prime}\right)$
Each non tree edge ( $v_{l, h}, v_{l, h}$ )
satisfies one of the following:

- ${ }^{\prime}=1$
- $l^{\prime}=l-1$ and $r<k$, where $v_{l-1, r}$ is the parent of $v_{l, h}$.



## APPROXIMATL RESULTS: OUTERPLANAR GRAPHS(6)

Th. $G=(V, E) ;$ OBFT $T=\left(V, E^{\prime}\right):$

- If $\left(v_{l, h}, v_{l, k}\right), h<k$, then $k=h+1$
- If $v_{l, h}$ is a child of $v_{l-1, i}$ and $\left(v_{l, h}, v_{l-1, k}\right)$ is a non tree edge, $i<k$, then $k=i+1$ and $v_{l, h}$ is the rightmost of its siblings
(orange edges are not admissible)



## APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(8)

- Lemma. If the root of the tree has an already assigned color, then $\Delta+3$ colors are necessary.


General


## APPROXIMATE RESUITS: OUTERPLANAR GRAPHS(7)

Lemma. If $\Delta \geq 4, \Delta+2$ colors are necessary.
Proof.

$\begin{array}{lllllll}0 & 1 & 2 & 3 & 4 & 5 & 6\end{array}$

## APPROXIMATL RESULTS: OUTERPLANAR GRAPHS(9)

- INPUT: G outerplanar with max degree $\Delta$
- OUTPUT: a feasible $L(2,1)$-labeling for $G$
- Consider a node $v$ with max degree and run an OBFS starting from $v$
- Label $(v) \leftarrow 0$
- Label the first layer according to the previous Lemma
- For each layer $I \geq 2$, top-down, from left to right repeat

Label the children of $v_{l, k}$ according to the previous Lemma, eliminating from the palette the forbidden colors

## APPROXIMATL RESULTS: OUTERPLANAR GRAPHS(10)

An example:
T. G outerplanar with max degree $\triangle$

- Consider a node $v$ with max degree and run an OBFS starting from $v$
- Label $(v) \leftarrow 0$
- Label the first layer according to the previous Lemma
- For eac

Label the children of $v_{l k}$ according to the previous Lemm eliminating from the palette the forbidden colors

$\begin{array}{llllllll}0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\end{array}$
(only for the right child...)

## APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(12)

```
- a. one tree-edge to the father of v}\mp@subsup{v}{l-1,k}{
- b at most three non-tree edges connecting v}\mp@subsup{v}{l-1,k}{}\mathrm{ with some nodes either
at the same layer or at the previous layer
- c. at most two non-tree edges from the leftmost sibling
-d. at most two non-tree edges from the rightmost sibling
```

The other outgoing edges do not contribute to the labeling during this
step.


## APPROXIMATZ RESULTS: OUTERPLANAR GRAPHS(11)

## Correctness and Bounds

- Best previously known results: $\lambda \leq \Delta+8$ for each $\Delta$ [BKTL’00]
- Conjecture [BKTLOO]: $\lambda \leq \Delta+2$ for each $\Delta$
- Th. The provided algorithm correctly $L(2,1)$-labels each outerplanar graph with max degree $\Delta \geq 8$ with $\lambda \leq \Delta+2$ in linear time; otherwise, at most 11 colors are anyway necessary.
- Proof. By induction, considering the edges coming out from the subgraph induced by any node and its children..


## APPROXIMATZ RESULTS: OUTERPLANAR GRAPHS(13)

(Proof sketch) By inductive hypothesis at most $\Delta+3$ colors have been used. We prove that they are sufficient to label the children of $v_{l-1, k}$.
We cannot use:

- At most 3 colors (for $v_{l-1, k}$ ) $\rightarrow \Delta$
- At most 1 color (edge a) $\rightarrow \Delta-1$
- At most $x$ colors (edges b), $0 \leq x \leq 3 \rightarrow \Delta-1-x$ for the $\Delta-1-x$ children of $v_{l-1, k}$
- The c edges give conditions only on 1 or 2 nodes, hence it is possible to arrange
- Analogously for the d edges



## APPROXIMATZ RESULTS: OUTERPLANAR GRAPHS(14)

The special case $\Delta=3$

- There exists an infinite class of outerplanar graphs having $\Delta=3$ requiring $\lambda=\Delta+3$
- It is possible to provide a labeling algorithm for these graphs using $\lambda \leq \Lambda+5$



## VARIATIONS OF THE PROBLEM

## APPROXIMATZ RESULTS: OUTERPLANAR GRAPHS(15)



## VARIATIONS OF THE PROBLEM (1) <br> ORIENTED L(2,1)-LABELIIG

- An oriented $L(2,1)$-labeling of a directed graph $G$ is a function assigning colors from $0, \ldots, \lambda$ to the nodes of $G$ so that nodes at distance 2 in the graph take different colors and adjacent nodes take colors at distance 2.
- Oriented $L(2,1)$-labeling problem minimizing $\lambda$
- Note. The minimum value of $\lambda$ can be very different from the value of the same parameter in the undirected case. Example: trees...


## VARIATIONS OF THE PROBLEM (2)

ORIENTED L( 2,1 )-LABELING

- Reminder: In undirected trees, $\Delta+1 \leq \lambda \leq \Delta+2$, and the exact value is linearly decidible
[Chang \& Kuo ‘96, Hasunama et al. 2008]
- In directed trees, $\lambda \leq 4$
[Chang $\underset{0}{\&}$ Liaw '03]



## VARIATIONS OF THE PROBLEM (4) bactibone coloring

If the topology has a backbone, where the transmitting power is higher wrt the rest of the network:

- A Backbone coloring of a graph $G$ wrt a graph $H$ is a function assigning integer values to the nodes of the graph such that:
$|I(u)-l(v)| \geq 2$ if $(u, v)$ is an edge of $H$ and
$|I(u)-I(v)| \geq 1$ if $(u, v)$ is an edge og G-H.
- Backbone coloring problem:
minimizing $\lambda$


## VARIATIONS OF THE PROBLEM (3)

$L\left(h_{b}, \ldots, h_{I}\right)$ Labrung
With the aim of making the model more realistic:

- An $L\left(h_{1}, \ldots, h_{k}\right)$-labeling of a graph $G$ is a function assigning integer values to the nodes of the graph such that:
$|l(u)-I(v)| \geq h_{i}$ if $u$ and $v$ are at distance $i$ in the graph, $1 \leq i \leq k$.
- $L\left(h_{1}, \ldots, h_{k}\right)$-labeling problem: minimizing $\lambda$
- Particularly interesting: $L(2,1,1)$ and $L(\delta, 1, \ldots, 1)$.
- Even these special cases are NP-hard on general graphs, so special classes of graphs are handled.


## VARIATIONS OF THE PROBLEM (5) <br> $n$ MUUZTPLEL LI $h, A$-LABELING

In practice, each transmitting station is able to handle more than one channel, so a set of channels is assigned to it.

- Given two set of integer values / and J, we define:

$$
\operatorname{dist}(I, J)=\min \{|i-j|: i \text { in } I \text { and } j \text { in } J\}
$$

## Example:

$I=\{0,1,2\} ; J=\{4,5,6\} ; \operatorname{dist}\{I, J\}=2$.

## VARIATIONS OF THE PROBLEM (6)

7-MUUTIPLE $L(h, A, A-L A B E L I N G$

- An n-multiple $L(h, k)$-labeling of a graph $G$ is a function assigning $n$ integer values to each node of the graph so that:
$\operatorname{dist}(I(u), I(v)) \geq h$ if $(u, v)$ is an edge of $G$ and
$\operatorname{dist}(l(u), l(v)) \geq k$ if $u$ and $v$ are at dist. 2 in $G$.
- n-multiple L(h,k)-labeling problem:
minimizing $\lambda$, given $n$.


## VARIATIONS OF THE PROBLEM (8)

freouency hssignuent in a gsin neiwork

In a GSM network, the cells need to use different frequences, in order not to interfere.

- Coloring map problem: given a plane map, the problem consists in coloring each region in such a way that adjacent regions take different colors and that the min number of colors is used.
- Four Color Theorem: It is always possible to color a map using at most 4 colors.

In the special case in which the network is a GSM:

- The network is a cellular network with hexagonal cells.
- Each cell has its own station connecting the fixed network devices with the mobile devices that are at moment inside the cell.
- Mobile phones connect to the GSM network trying to communicate with the station associated to the cell where they lie.


## VARIATIONS OF THE PROBLEM (9)

FREDUENCY ASSIGNMENT IN A GSII NETWORK

- It follows that 4 different frequencies are sufficient for an arbitrary GSM network:



## VARIATIONS OF THE PROBLEM (10)

frequency hissigninen in a gsin neiwork

- In fact, more sophisticated variations of this problem lead to 7 colors -> L(h,k)-labeling again

More in detail...


## VARIATIONS OF THE PROBLEM (12)

frequency hssigninit in a gim nerwork

- In the 3rd generation of mobile systems, the introduction of CDMA (Code Division Multiple Access) has enabled the reuse of the whole frequency band in each cell: instead of dividing the signal space in time or frequency, a code of pseudorandom sequence is used to differentiate the signal from each transmitter.
- In this context, the labeling schemes were of much reduced importance.


## VARIATIONS OF THE PROBLEM (11)

FREQUENCY ASSIGNMENT IN A GSIM NETWORK

- In wireless communication networks of the 1st and 2nd generation, the concept of cellular channel allocation and spatial frequency reuse were the key ideas that have driven the initial success of mobile telephony.
- For example, a seven color labeling of a hexagonal grid was on the basis of the AMPS (American Mobile Phone System).
- The same scheme existed for GSM.
"...


## VARIATIONS OF THE PROBLEM (13)

FREQUENCY ASSIGNMENT IN A GSII NETWORK

- The 4th generation mobile standards mainly use Orthogonal Frequency Division Multiple Access schemes.
- These schemes divide the signal space in time slots and orthogonal frequencies.
- At the middle of a cell, all slots of time and frequencies are allocated to users. At the edge of a cell, only part of the band is used and a three color scheme is used.
- Even this model can be reduced to a labeling scheme. For further details:


## A PRRENTHESIS ON THE 4 COLOR PROBLEM (1)

- Given a map, it can be naturally considered as a planar graph $G$.
- Given $G$, let $G^{*}$ its dual graph:
- Put a node of $G^{*}$ in each region of $G$
- Connect two nodes of $G^{*}$ iff the corresponding regions (faces) are adjacent (i.e. share an edge in $G$ )
- A vertex coloring of $G^{*}$ corresponds to a map coloring of $G$.


## a PRRENTHESIS ON THE 4 COLOR PROBLEM (3)

There are some interesting results for other numbers of colors:

- 2-coloring.

Polynomially solvable:

- Assign a color to a region.
- Assign the other color to its neighbor regions.
- Assign the first color to its neighbor regions.
- Continue until the regions have been all colored or there is a color conflict. In this latter case the map is not 2colorable.


## A PARENTHESIS ON THE 4 COLOR PROBLEM (2)

- In fact, cartographers have always known that 4 colors were enough for each kind of map, but in 1852 Francis Guthrie wondered whether this fact could be proved.
- After more than 100 years, and many (wrong) announcements, Appel and Haken proved the 4 Color Theorem in 1976.
-The complete proof is computer assisted because it exhaustively examines more than 1700 configurations.
- More recently, Robertson, Sanders, Seymour, and Thomas wrote a new proof, needing to examine "only" 633 configurations.


## I PARENTHESIS ON THE 4 COLOR PROBLEM (4)

- 3-coloring
- NP-hard, hence no algorithms to decide whether a map is 3-colorable or not.
- Method: exhaustively try all the color combinations for the regions.
- Inapplicable: for $N$ regions, there are $3^{N}$ possibilities. (if $N=48$ the combinations are about $8 \times 10^{22}$ )
- ...


## A PRRENTHESIS ON THE 4 COLOR PROBLEM (5)

- 3-coloring (cntd)
- There are some techniques in order to simplify the map before coloring it (for example, if a region has only 2 neighbor regions, it can be eliminated from the map: when it is re-inserted, it will be colored with the third color) but the worst case time complexity is the same.


## A PARENTHESIS ON THE 4 COLOR PROBLEM (7)

## - 5-coloring

- It is relatively easy to color a map using 5 colors. There is an algorithm that first simplifies the map eliminating all the regions and then re-insert them assigning the correct color.


## A PRRENTHESIS ON THE 4 COLOR PROBLEM (6)

## - 4-coloring

- The proof of the 4 color theorem is constructive, and so it shows how to find a feasible coloring, but the number of cases is too high to be useful in practice.
- There are some transformations, similar to those used for the 3-coloring, but they do not eliminate the need of exhaustively try all the possibilities.


## A PARENTHESIS ON THE 4 COLOR PROBLEM (8)

We conclude with a puzzle:
Try to 4-color these 2 maps...


In 1975 Martin Gardner claimed
he could prove that this map was not 4-colorable (April fool)

## A PARENTHESIS ON THE 4 COLOR PROBLEM (9)

Solutions


