SECOND PART: WIRELESS NETWORKS

2.A. AD HOC NETWORKS

THE FREQUENCY ASSIGNMENT PROBLEM I.E. THE *L(h,k)*-LABELING OF GRAPHS

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Network Algorithms
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THE PROBLEM

THE PROBLEM (1)

- There is not "the" frequency assignment problem.
- Frequency assignment is necessary in many different types of wireless networks.
- Depending on the particular network, the understanding of frequency assignment varies.
- For this reason, several "flavors" of frequency assignment are present in the literature.

THE PROBLEM (2)

- Wireless communication between two points is established with the use of a transmitter and a receiver.
- The transmitter generates electrical oscillations at a radio frequency.
- The receiver detects these oscillations and transforms them into sounds or images.
- When two transmitters use the same frequency, they may interfere.

THE PROBLEM (3)

One of the most popular applications of wireless communication is the establishment of fixed cellular telecommunication networks.

In contrast to mobile cellular networks, in non-mobile or fixed systems both the transmitters and the receivers are located at fixed points in the area of interest.

Fixed cellular networks provide a financially attractive alternative to the construction of conventional wired networks.

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THE PROBLEM (4)

Moreover, the introduction of new services, such as data communication (internet, e-mail) and video-conferencing cause shortage of capacity in existing wired networks.

Point-to-point wireless connections can be used as an alternative to the extension of the capacity of these wired networks.

In both cases no cable connections have to be established.

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THE PROBLEM (5)

A disadvantage of point-to-point connections is that the transmitter and receiver have *to see* each other, which means that there should be no obstacles in between them.

As a consequence, transmitters and receivers have to be built at high locations (e.g., at the roof of apartment and office buildings).

Although the transmitters are directed to the receivers, their signals can interfere. Especially if signals cross each other, the use of (almost) the same frequencies should be avoided.

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THE PROBLEM (6)

Another application that has similarities with fixed cellular networks stems from the military.

In military communication networks, wireless connections have to be established between pairs of transceivers.

These connections, or radio links, can interfere with each other, if they use similar frequencies in the same area.

THE PROBLEM (7)

The rapid development of new wireless services (e.g. digital cellular phone networks) resulted in a run out of the most important (and expensive) resource: frequencies in the radio spectrum.

Like with all scarcely available resources, the cost of frequency-use provides the need for economic-use of the available frequencies.

Reuse of frequencies within a wireless communication network can offer considerable economies.

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THE PROBLEM (8)

However, reuse of frequencies may also lead to a loss of quality of communication links.

Namely, the use of (almost) the same frequency for multiple wireless connections can cause an interference between the signals that is unacceptable.

A solution to the frequency assignment problem balances the economies of reuse of frequencies and the loss of quality in the network.

Quantification of the different aspects results in a mathematical optimization problem.

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THE PROBLEM (9)

In general, Frequency Assignment Problems (FAPs) have two basic aspects:

1. a set of wireless communication connections must be assigned frequencies such that, for every connection, data transmission between the transmitter and receiver is possible. The frequencies should be selected from a given set that may depend on the location.
2. The frequencies assigned to two connections may incur interference resulting in a loss of quality of the signal.

But, what interference is?

THE PROBLEM (10)

Two conditions must be fulfilled in order to have interference of two signals:

a) The two frequencies must be close on the electromagnetic band (Doppler effects) or (close to) harmonics of one another.

   The latter effect is limited, since the frequency bands from which we can choose are usually so small that they do not contain harmonics.

b) The connections must be geographically close to each other.

THE PROBLEM (11)

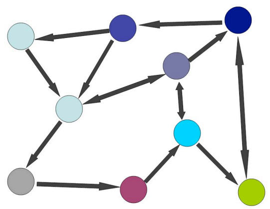
Both aspects are modeled in many different ways in the literature.

Hence: various models.

They differ in the types of constraints and in the objectives to be optimized.

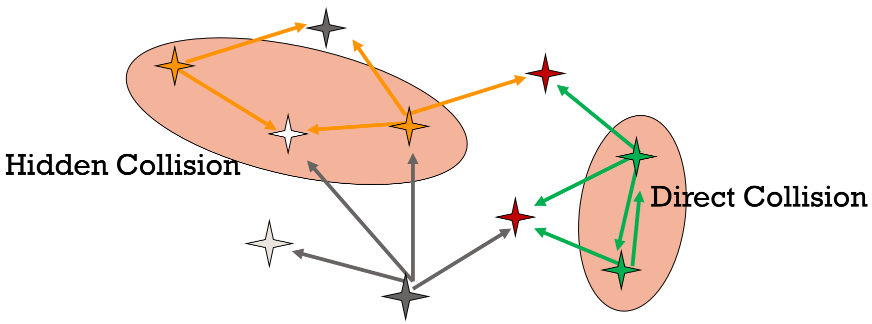
Here we describe a simplified model.

THE GRAPH MODEL



THE MODEL (1)

In our model:



THE MODEL (2)

Interference due to re-use of the same channel at "close" or "very close" sites.

Contributions also from sites using only close channels, since in practice transceivers do not operate exclusively within the frequencies of their assigned channels.

Direct Collisions: stations positioned in close locations receive channels at least h apart

Hidden Collisions: channels for stations positioned in very close locations are at least k apart

$L(h,k)$-Labeling Problem

L(*h*,*k*)-LABELING (1)

Interference Graph

- One node per station
- One edge between two stations if they may communicate (and hence interfere)
- Labels interpreted as channels assigned to the nodes.

f: node coloring function s.t.

- $\forall\ u, v \in V\ |\mathbf{f}(u) - \mathbf{f}(v)| \geq \mathbf{h}$ if $(u,v) \in E$
- $\forall\ u, v \in V\ |\mathbf{f}(u) - \mathbf{f}(v)| \geq \mathbf{k}$ if $\exists\ w \in V$ *s.t.* $(u,w) \in E$ and $(w,v) \in E$

Objective: minimizing the *bandwidth* $\boldsymbol{\sigma}_{h,k}$

Minimum bandwidth: $\boldsymbol{\lambda}_{h,k}$

L(*h*,*k*)-LABELING (2)

Obs. The condition:

- $\forall\ u, v \in V\ |\mathbf{f}(u) - \mathbf{f}(v)| \geq k$ if $\exists\ w \in V$ *s.t.* $(u,w) \in E$ and $(w,v) \in E$

is often written as:

- $\forall\ u, v \in V\ |\mathbf{f}(u) - \mathbf{f}(v)| \geq k$ if dist$(u,v)=2$

The first one works both when $\mathbf{h} \geq \mathbf{k}$ and when $\mathbf{h} < \mathbf{k}$.

It allows a triangle to be labeled with colors at mutual distance at least *max*{$\mathbf{h}$,$\mathbf{k}$}, even if its nodes are at distance 1.

When $\mathbf{h} \geq \mathbf{k}$ the two conditions coincide.

Example: $L(1,2)$

L(*h*,*k*)-LABELING (3)

Usually, the minimum used color is *0*.

So, an $L(h,k)$-labeling having span $\sigma_{h,k}(G)$ uses $\sigma_{h,k}(G)+1$ different colors.

This is slightly counter-intuitive, but it is used for historical reasons.

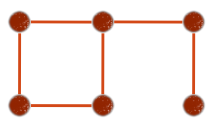
The problem has been introduced in the '90s with $h=2$ and $k=1$ in relation with a frequency assignment problem [Griggs e Yeh '92, Robertson '91]

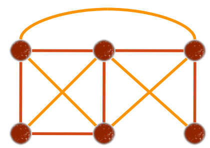
This problem was already known in combinatorics in the case $h=1$ and $k=1$ (coloring the square of a graph) [Wegner '77]

A PARENTHESIS ON THE L(*1*,*1*)-LABELING (1)

When h=*1* and k=*1* the problem is equivalent to the classical vertex coloring of the square of a graph.

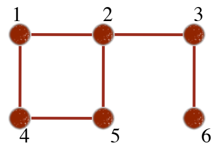
Given a graph $G=(V,E)$, its square G^2 is defined as a graph having node set equal to V, and an edge between u and v is in G^2 iff:
- either (u,v) is in E
- or u and v are connected by a length 2 path in G

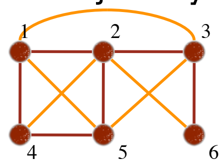




A PARENTHESIS ON THE L(*1*,*1*)-LABELING (2)

If the graph is stored in an adjacency matrix:





Compute A(G)^2:

$(a_{ij})^2 = \Sigma_{k=1\ldots n}\, a_{ik}\, a_{kj}$

$$A(G)=\begin{array}{c}0\,1\,0\,1\,0\,0\\1\,0\,1\,0\,1\,0\\0\,1\,0\,0\,0\,1\\1\,0\,0\,0\,1\,0\\0\,1\,0\,1\,0\,0\\0\,0\,1\,0\,0\,0\end{array}\qquad A(G^2)=\begin{array}{c}0\,1\,1\,1\,1\,0\\1\,0\,1\,1\,1\,1\\1\,1\,0\,0\,1\,1\\1\,1\,0\,0\,1\,0\\1\,1\,1\,1\,0\,0\\0\,1\,1\,0\,0\,0\end{array}\qquad A(G)^2=\begin{array}{c}2\,0\,1\,0\,2\,0\\0\,3\,0\,2\,0\,1\\1\,0\,2\,0\,1\,0\\0\,2\,0\,2\,0\,0\\2\,0\,1\,0\,2\,0\\0\,1\,0\,0\,0\,1\end{array}$$

$(a_{ij})^2=x$ iff there are x 2-length paths between i and j.
To store together the knowledge about 1-and 2-length paths:
$A(G)^2+A(G) \rightarrow A(G^2)$

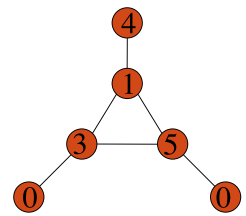
L(*h*,*k*)-LABELING (4)

After its definition, the $L(h,k)$-labeling problem has been used to model several problems:
- a kind of integer 'control code' assignment in packet radio networks to avoid hidden collisions ($L(0,1)$-labeling problem)
- channel assignment in optical cluster-based networks ($L(0,1)$- or $L(1,1)$-labeling depending on the fact that the clusters can contain one ore more nodes)
- more in general, channel assignment problems, with a channel defined as a frequency, a time slot, a control code, etc.

$L(h,k)$-labeling has been studied following many different approaches: graph theory and combinatorics, simulated annealing, genetic algorithms, tabu search, neural networks,

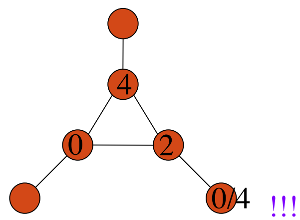
L(*h*,*k*)-LABELING (5)

Example: $L(2,1)$-labeling of:



$\lambda_{2,1}\leq 5$

$\lambda_{2,1}=5$: by contradiction



L(h,k)-LABELING (6)

Lemma: $\lambda_{dh,dk}=d\,\lambda_{h,k}$

Proof. Divided into two parts: $\lambda_{dh,dk}\geq d\,\lambda_{h,k}$ and $\lambda_{dh,dk}\leq d\,\lambda_{h,k}$.

1. $\lambda_{dh,dk}\geq d\,\lambda_{h,k}$

Let f an $L(dh, dk)$-labeling. Define $f'=f/d$.

f' is an $L(h,k)$-labeling and $\lambda_{dh,dk}/d = \sigma_{h,k}(f')\geq \lambda_{h,k}$.

2. $\lambda_{dh,dk}\leq d\,\lambda_{h,k}$

Similarly, let f an $L(h,k)$-labeling. Define $f'=fd$.

f' is an $L(dh, dk)$-labeling and $\lambda_{dh,dk}\leq \sigma_{dh,dk}(f')=d\,\lambda_{h,k}$. ■

It follows that we can restrict to use values of h and k mutually prime.

L(h,k)-LABELING (7)

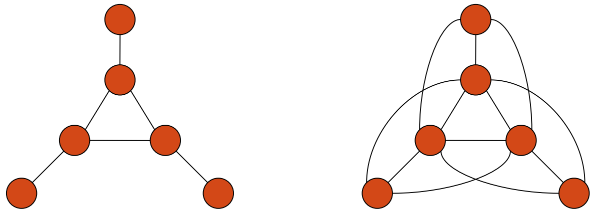
PROBLEM: What if $f'=f/d$ does not produce integer values?

Lemma: Let $x, y \geq 0$, $d>0$ and k in $\mathbb{Z}^+$.
If $|x-y|\geq kd$, then $|x'-y'|\geq kd$,
where $x'=\lfloor x/d\rfloor d$ and $y'=\lfloor y/d\rfloor d$

It follows we can restrict to use values of h and k integer and mutually prime.

L(h,k)-LABELING (8)

- The case $k=0$, for any h, is not usually considered as an $L(h,k)$-labeling problem, as it coincides with the classical vertex coloring
- The case $h=k$ is very studied in the literatue as the vertex coloring of the square of a graph.
- The case $h=2k$ is the most studied.



NP-COMPLETENESS RESULTS (1)

The decisional version of the problem is NP-complete, even for small values of h and k:

$L(0,1)$-labeling of planar graphs

[Bertossi, Bonuccelli '95]

$L(1,1)$-labeling of general, planar, bounded degree and e unit-disk graphs

[McCormick '83], [Ramanathan, Loyd '92], [Ramanathan '93], [Sen, Huson '97]

NP-COMPLETENESS RESULTS (2)

Th. The $L(2,1)$-labeling problem on diam. 2 graphs is NP-complete [Griggs, Yeh '92]

Proof. Consider the following special form of the decisional problem:

DL. Instance: $G=(V,E)$ diam. 2 graph

Question: $\lambda_{2,1}(G) \leq |V|$?

IDL. Instance: $G=(V,E)$

Question: Does exist an f injective s.t.

$|f(x)-f(y)| \geq 2$ if $(x,y) \in E$

and its codomain is $\{0, \ldots, |V|-1\}$?

it never maps distinct elements of its domain to the same element of its codomain

NP-COMPLETENESS RESULTS (3)

(proof of NP-completeness cntd)

Finding a solution for IDL means finding a Hamiltonian path in G^C:

Since f is injective, f^{-1} is defined.

Give an order to nodes:

$v_i = f^{-1}(i),\ 0 \leq i \leq |V|-1$

Observe that, since v_i and v_{i+1} cannot be adjacent in G, they are adjacent in G^C, hence

$v_0, v_1, \ldots, v_{|V|-1}$ is a Hamiltonian path.

NP-COMPLETENESS RESULTS (4)

(proof of NP-completeness cntd)

Even the reverse holds:

Given a Hamiltonian path in G^C

$v_0, v_1, \ldots, v_{|V|-1}$ define f such that $f(v_i)=i$, $0 \leq i \leq |V|-1$.

f is trivially injective; furthermore, given an edge $\{x,y\}$ of G, $x=v_i$ and $y=v_j$, it must hold:

$|f(x)-f(y)| \geq 2$ because x and y are not adjacent in G^C.

It follows that the two problems are equivalent.

NP-COMPLETENESS RESULTS (5)

(proof of NP-completeness cntd)

The following problem:

HP. Instance: $G=(V,E)$

Question: Does G have a hamiltonian path?

is NP-complete, so even IDL is NP-complete.

Instance: $G=(V,E)$ diam. 2 graph

Question: $\lambda_{2,1}(G) \leq |V|$?

DL is in NP:

We can verify in polynomial time that G has diameter 2, whether a labeling f is a feasible $L(2,1)$-labeling, and whether $\lambda_{2,1}(G) \leq ||f(G)|| \leq |V|$.

NP-COMPLETENESS RESULTS (6)

(proof of NP-completeness cntd)

Transformation from IDL to DL to prove that DL is NP-complete:

Given an instance of IDL G, construct G':

- $V'=V \cup \{x\}$
- $E'=E \cup \{\{x,a\}$ for each a in $V\}$

So $|V'|=|V|+1$ and G' has diameter 2

NP-COMPLETENESS RESULTS (7)

(proof of NP-completeness cntd)

We prove that from a solution for DL it is possible to deduce a solution for IDL, i.e. there is an injection f s.t.

$|f(x)-f(y)|\geq 2$ for every $(x,y) \in E$ iff $\lambda_{2,1}(G')\leq|V'|$.

- => If there exists an injection f defined on V that satisfies the condition above, define $g(v)=f(v)$ for all $v\in V$ and $g(x)=|V|+1=|V'|$.

Easily g is an $L(2,1)$-labeling for G' and $\lambda_{2,1}(G')$ $\leq||g(G')||\leq|V'|$

NP-COMPLETENESS RESULTS (8)

(proof of NP-completeness cntd)

- <= Conversely, suppose that $\lambda_{2,1}(G')\leq|V'|$, i.e. there exists a feasible $L(2,1)$-labeling g s.t. $||g(V')||\leq|V'|=|V|+1$.

Observe that G' of diam. 2 implies that

$g(a)\neq g(b)$ for each $a\neq b$

- Suppose $g(x)\neq|V|+1$ and $\neq 0$. By the property of $L(2,1)$-labeling, there is no v in V such that $g(v)=g(x)-1$ or $g(x)+1$. So we need $|V|+3$ labels for V' i.e. $\lambda_{2,1}(G')\geq|V'|+1$: a contardiction.

NP-COMPLETENESS RESULTS (9)

(proof of NP-completeness cntd)

- So $g(x)$ is either 0 or $|V|+1$.
  - If $g(x)=|V|+1$ => $f(v)=g(v)$ OK
  - If $g(x)=0$ => $f(v)=g(v)-2$ OK

In any case, there exists f injective s.t. its codomain is $\{0, \ldots, |V|-1\}$.

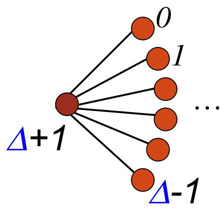
The NP-compleness of DL follows. ■

Literature in different directions:

- Lower and upper bounds for $\lambda_{h,k}$
- Limitation to special graph classes:
  - Exact labelings
  - Approximate labelings

LOWER BOUNDS (1)

- $\lambda_{2,1} \geq \Delta+1 \;=\; (\Delta-1)1+2$
- $\lambda_{h,k} \geq (\Delta-1)k+h$ for $h \geq k$

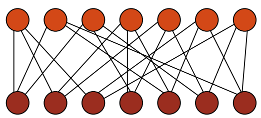


LOWER BOUNDS (2)

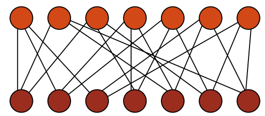
- $\exists\, G$ s.t. $\lambda_{2,1}(G) \geq \Delta^2 - \Delta$ [Griggs, Yeh '92]

Incidence graph of a projective plane $\pi(n)$ of order n, $G=(U \cup V, E)$ s.t.

- $|U|=|V|=n^2+n+1$
- $u \in U$ corresonds to a point P_u of $\pi(n)$
- $v \in V$ corresonds to a line l_v of $\pi(n)$
- $E=\{(u,v)$ s.t. $P_u \in l_v\}$



LOWER BOUNDS (3)



- G is regular and $\Delta=n+1$
- For each pair of nodes in U (or in V), their distance is 2,
- $\forall\, u,v \in U$ ($\in V$), $|Adj(u) \cap Adj(v)|=1$

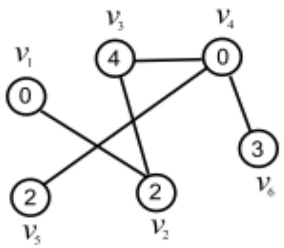
$\Rightarrow \lambda_{2,1}(G) \geq |U|-1=|V|-1=\Delta^2-\Delta$

UPPER BOUNDS (1)

Greedy Algorithm:

Given a graph G with nodes $v_1, v_2, \ldots, v_n$,

label its nodes in order assigning to v_i the smallest color not conflicting with the labels of its neighborhood (dist. 1 and 2)



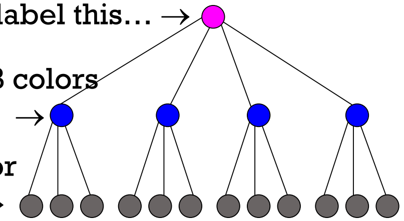
UPPER BOUNDS (2)

- Th. $\lambda_{2,1}(G) \leq \Delta^2+2\Delta$ [Griggs, Yeh '92]
- Proof.

In order to label this… →

…we have to eliminate at most 3 colors for each one of these… →

…and at most one color for each one of these… →

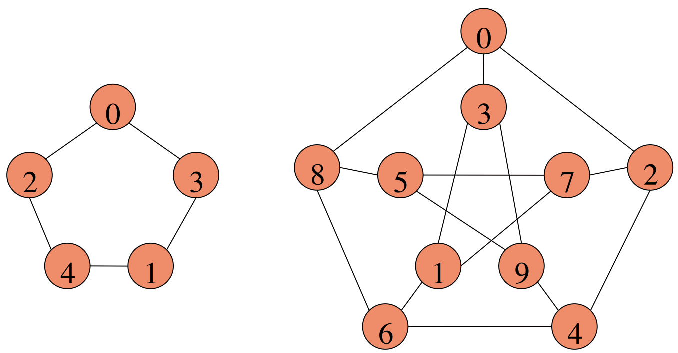


We can label all the graph with at most $1+3\Delta+(\Delta-1)\Delta$ colors.

UPPER BOUNDS (3)

Conjecture: $\lambda_{2,1}(G) \leq \Delta^2$ [Griggs, Yeh '92]

This upper bound is tight: some graphs with degree Δ, diameter 2 and Δ^2+1 nodes have λ at leats Δ^2.



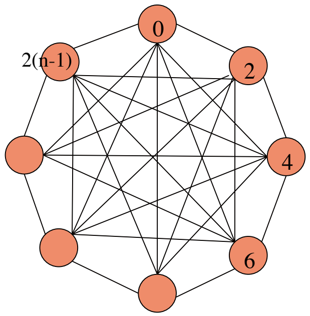
UPPER BOUNDS (4)

Conjecture: $\lambda_{2,1}(G) \leq \Delta^2$ [Griggs, Yeh '92]

- $\lambda_{2,1}(G) \leq \Delta^2+2\Delta-4$ [Jonas '93]
- $\lambda_{2,1}(G) \leq \Delta^2+\Delta$ [Chang, Kuo '96]
- $\lambda_{2,1}(G) \leq \Delta^2+\Delta-1$ [Kral, Skrekovski '03]
- $\lambda_{2,1}(G) \leq \Delta^2+\Delta-2$ [Goncalves '05]
- $\lambda_{2,1}(G) \leq \Delta^2$ for sufficiently large values of Δ [Havet, Reed and Sereni '08]

EXACT RESULTS: CLIQUES K_n

- $\lambda_{2,1}(K_n)=2(n-1)$
- All nodes are pairwise adjacent

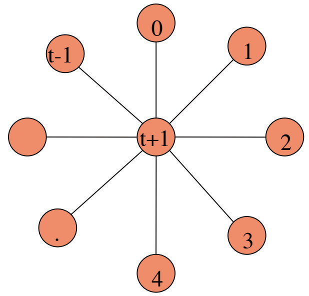


EXACT RESULTS: STARS $K_{1,t}$

- $\lambda_{2,1}(K_{1,t})=t+1$

Proof.

- $\lambda_{2,1}(K_{1,t})\leq t+1$ easy
- $\lambda_{2,1}(K_{1,t})\geq t+1$
  by contradiction



EXACT RESULTS: TREES T_n (1)

- $\lambda_{2,1}(T_n)=\Delta+1$ or $\Delta+2$

Proof.

- $\lambda_{2,1}(T_n)\geq\Delta+1$ because T_n contains a $K_{1,\Delta}$
- $\lambda_{2,1}(T_n)\leq\Delta+2$
  first-fit (greedy) labeling:

Order the nodes of T_n: $T_{n-1}=T_n-\{v_n\}$ where v_n is a leaf.
In general $T_i=T_{i+1}-\{v_{i+1}\}$

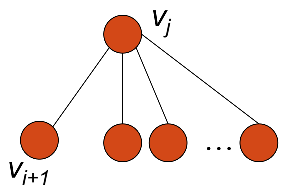
Label v_1 with *0.*

Label v_i with the first available color.

EXACT RESULTS: TREES T_n (2)

(proof: either $\lambda_{2,1}(T_n)=\Delta+1$ or $\Delta+2$ – cntd)

Assume we have already labeled all nodes from v_1 to v_i and we are going to label v_{i+1}:



v_j parent of v_{i+1} necessarily $j\leq i+1$ (the nodes that are closer to the leaves have larger numebring)

v_j has at most $\Delta-1$ further adjacent nodes

At most 3 colors are forbidden due to v_j
At most $\Delta-1$ colors are forbidden due to the nodes that are adjacent to v_j
If we have at least $(\Delta-1)+3+1$ colors, we are always able to label v_{i+1} i.e. $\lambda_{2,1}(T_n)\leq\Delta+2.$ ■

EXACT RESULTS: TREES T_n (3)

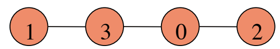
- This proof has been proposed by Griggs e Yeh ['92], who have also conjectured that it is NP-complete to decide whether the correct value is $\Delta+1$ or $\Delta+2$.
- Chang e Kuo ['96] have disproved this conjecture by providing a polynomial algorithm based on the dynamic programming technique and having time complexity $O(\Delta^{4.5}\, n)$.
- Many authors have proposed many other algorithms aiming at improving the time complexity.
- Finally, Hasunama, Ishi, Ono, Uno ['08] have proposed a linear algorithm.

EXACT RESULTS: PATHS P_n

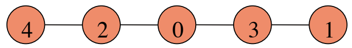
- $\lambda_{2,1}(P_2)=2$
- $\lambda_{2,1}(P_3)=3$

From the results for the stars

- $\lambda_{2,1}(P_4)=3$

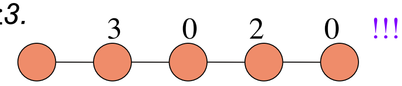


- $\lambda_{2,1}(P_n)=4$ if $n\geq 5$



To prove that $\lambda_{2,1}(P_5)\leq 4$:

P_5 includes a P_4 so $\lambda_{2,1}(P_5)\geq 3$.
By contradiction $\lambda_{2,1}(P_5)=3$

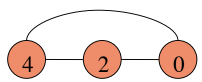


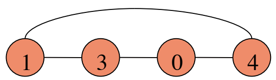
If $n\geq 5$ the result follows from the previous one and from the result for trees.

EXACT RESULTS: CYCLES C_n (1)

- $\lambda_{2,1}(C_n)=4$

If $n\leq 4$: case by case:





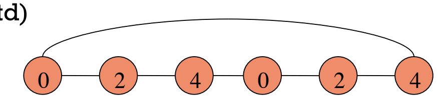
If $n\geq 5$: C_n contains P_n so $\lambda_{2,1}(C_n)\geq 4$.
It also holds $\lambda_{2,1}(C_n)\leq 4$:

3 cases: …

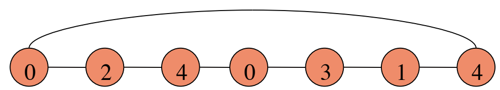
EXACT RESULTS: CYCLES C_n (2)

(proof: $\lambda_{2,1}(C_n)=4$ – cntd)

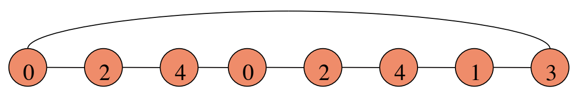
1. $n=0 \ (mod\ 3)$



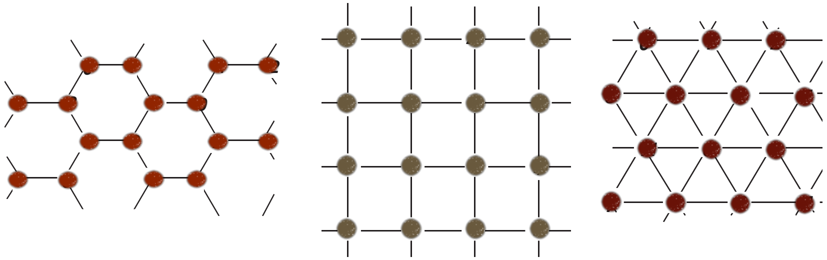
2. $n=1 \ (mod\ 3)$



3. $n=2 \ (mod\ 3)$

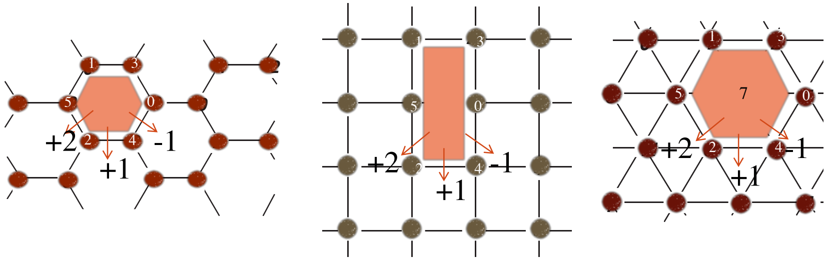


EXACT RESULTS: GRIDS (1)



EXACT RESULTS: GRIDS (2)

$\lambda_{2,1}(\Delta)=\Delta+2$



EXACT RESULTS: GRIDS (3)