Normal and Canonical Forms

Prof. Daniele Gorla

BFs vs BEs

Theorem: for every BE there exists a unique associated BF
Proof:

- Through perfect induction, we build the truth table
- this describes *the* associated BF

Q.E.D.

The converse does NOT hold: *for every BF there exist infinitely many equivalent BEs*

Example:

x	f
0	1
1	0

Bes that have this truth table are (among the others):

$$\overline{x}, \overline{x}+0, \overline{x}+0+0, \overline{x}+0+0+0, \ldots$$

We shall now define *canonical forms* such that every BF has one unique canonical BE associated.

From a BF to the (disjunctive) canonical form through examples

x	y	f
0	0	0
0	1	0
1	0	0
1	1	1

f holds 1 if and only if $x = 1$ and $y = 1$
i.e., if and only if $x \cdot y = 1$
Hence, $f = x \cdot y$

x	y	f
0	0	0
0	1	0
1	0	1
1	1	0

f holds 1 if and only if $x = 1$ and $y = 0$ (that is, , $\overline{y} = 1$)
i.e., if and only if $x \cdot \overline{y} = 1$
Hence, $f = x \cdot \overline{y}$

x	y	f
0	0	0
0	1	0
1	0	1
1	1	1

f holds 1 if and only if $x = 1$ & $y = 1$ or $x = 1$ & $y = 0$
i.e., if and only if $x \cdot y + x \cdot \overline{y} = 1$
Hence, $f = x \cdot y + x \cdot \overline{y}$

Disjunctive Canonical Form (or SOP)

Let's assume to have n variables $\{x_1, \ldots, x_n\}$:

Every occurence of a variable, either in simple form x_i or negated $\overline{x}_i$, is called *literal*.

A *minterm* is a product of n literals $l_1 \cdot \ldots \cdot l_n$ such that
$l_i \in \{x_i, \overline{x}_i\}$, for every $i \in \{1,..,n\}$

A *Disjunctive Canonical Form* (or *SOP canonical form)*, is a sum (or disjunction, hence the name) of pairwise distinct minterms.

Example (n=3):

$$x_1 x_2 x_3 + \overline{x}_1 x_2 x_3 + x_1 \overline{x}_2 \overline{x}_3 + \overline{x}_1 \overline{x}_2 \overline{x}_3$$

DCFs and BFs

Let f be a function in the n variables $\{x_1,\dots,x_n\}$:

A minterm m is an *implicant* of f if, for every $b_1\dots b_n \in \{0,1\}^n$,

$$m(b_1\dots b_n) = 1 \Rightarrow f(b_1\dots b_n) = 1$$

The DCF associated to f is the DCF that contains all and only the minterms that are implicants of f.

Ex.:

$x_3\ x_2\ x_1$	f
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	0
1 1 0	0
1 1 1	1

$\bar{x}_3\bar{x}_2\bar{x}_1$ isn't an implicant of f: $m(000)$=1 but $f(000)$=0

$\bar{x}_3x_2\bar{x}_1$ is an implicant: the only assignment that makes m=1 is 010 and $f(010)$=1

The implicants of f are: $\bar{x}_3x_2\bar{x}_1, \bar{x}_3x_2x_1, x_3x_2x_1$

Hence, the DCF of f is $\bar{x}_3x_2\bar{x}_1 + \bar{x}_3x_2x_1 + x_3x_2x_1$

Identifying minterms

SAPIENZA
Università di Roma
Dipartimento di Informatica

OBS: for every minterm m, there esists a unique n-tupla of bits that make it 1.

Ex.: $\bar{x}_3x_2\bar{x}_1$ holds 1 iff $x_3 = x_1 = 0$ and $x_2 = 1$

In general, the n-tupla can be obtained by giving 1 to the variables that occur simple in m and 0 to those that occur negated.

Hence, we can create a bijection between the 2^n minterms with $\{0,1\}^n$:

$$m \leftrightarrow b_1\dots b_n \quad \text{iff} \quad m(b_1\dots b_n) = 1$$

If m corresponds to $b_1\dots b_n$ in this bijection and $b_1\dots b_n$, seen as a natural number codified in binary with n bits, corresponds to the decimal number k, then m will be called m_k.

Ex.: $\bar{x}_3x_2\bar{x}_1$ corresponds to 010; having $010_2 = 2_{10}$, we shall call m_2 such a minterm.

From DCF to the BF and vice versa

SAPIENZA
Università di Roma
Dipartimento di Informatica

- Given a BF, the associated DCF can be obtained by taking all the rows where the BF holds 1 and by considering all the minterms associated to such rows.
- Given a DCF, the associated BF can be obtained by putting 1 in all rows whose binary strings correspond to the minterms in the DCF and 0 elsewhere.

Ex.:

$x_3\ x_2\ x_1$	f
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	1
1 1 1	1

DCF: $m_1+m_3+m_5+m_6+m_7$

From a BE to its DCF

SAPIENZA
Università di Roma
Dipartimento di Informatica

Let E be any BE.

1. Push all its negations directly on its variables (De Morgan) and delete double negations (involution)
2. Turn the resulting expression in SOP form, by using distributivity of · over +
3. Delete possible copies of the summands (idempotency) and the products that contain a literal and its negation (complement and annihilator)

We now have a *form disjunctive normal form* (or *SOP normal form*), i.e., a SOP whose summands are not minterms, in general.

4. Multiply every summand that does not contain x_i with $(x_i + \bar{x}_i)$ (neutral and complement)
5. Turn the resulting expression in SOP form, by using distributivity of · over +
6. Delete possible copies of summands (idempotency)

Example

$$E = (x_1 + x_2(\overline{x_3 + \bar{x}_1 x_4}))x_3 + \overline{\bar{x}_2 x_4}$$

1) $E = (x_1 + x_2(\bar{x}_3 \cdot \overline{\bar{x}_1 x_4}))x_3 + (\bar{\bar{x}}_2 + \bar{x}_4) = (x_1 + x_2\bar{x}_3(\bar{\bar{x}}_1 + \bar{x}_4))x_3 + (x_2 + \bar{x}_4)$

$= (x_1 + x_2\bar{x}_3(x_1 + \bar{x}_4))x_3 + x_2 + \bar{x}_4$

2) $= x_1x_3 + x_2\bar{x}_3(x_1 + \bar{x}_4)x_3 + x_2 + \bar{x}_4 = x_1x_3 + x_1x_2\bar{x}_3x_3 + x_2\bar{x}_3x_3\bar{x}_4 + x_2 + \bar{x}_4$

3) $= x_1x_3 + x_2 + \bar{x}_4 \qquad \rightarrow \qquad$ *SOP Normal Form*

4) $= x_1x_3(x_2 + \bar{x}_2)(x_4 + \bar{x}_4) + x_2(x_1 + \bar{x}_1)(x_3 + \bar{x}_3)(x_4 + \bar{x}_4) + \bar{x}_4(x_1 + \bar{x}_1)(x_2 + \bar{x}_2)(x_3 + \bar{x}_3)$

5) $= m_{15} + m_{11} + m_{14} + m_{10} + m_{15} + m_{14} + m_{13} + m_{12} + m_7 + m_6 + m_5 + m_4 + m_{14} + m_{12} + m_{10} + m_8 + m_6 + m_4 + m_2 + m_0$

6) $= m_{15} + m_{14} + m_{13} + m_{12} + m_{11} + m_{10} + m_8 + m_7 + m_6 + m_5 + m_4 + m_2 + m_0$

$\rightarrow \qquad$ *SOP Canonical Form*

From a DNF to the BF

Also *(disjunctive) normal forms* can be used to quickly derive the truth table of a BF:

Let $\{x_1,\ldots,x_n\}$ be the variables in the DNF

Whereas every minterm univoquely identifies one single row of the truth table, every summand in a DNF (that is a product of literals) identifies a set of rows in the following way:

- If x_j appears negated, x_j must hold 0;
- If x_j appears simple, x_j must hold 1;
- If x_j doesn't appear, it can hold both 0 and 1.

Hence, we build the table by putting 1 in all rows that are identified by at least one summand of the DNF.

Example

$$E = \bar{x}_2 x_1 + x_3 x_2 x_1$$

The first summand holds 1 with the assignments 001 and 101
The second summand holds 1 only for 111 (it is a minterm!)
Hence,

$\bar{x}_2 x_1$

$x_3 x_2 x_1$

$x_3\ x_2\ x_1$	f
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

POS Forms (through examples)

$x\ y$	f
0 0	0
0 1	1
1 0	1
1 1	1

$f = \bar{x}y + x\bar{y} + xy$. But we can also describe f through its 0s

f holds 0 iff $x = 0$ and $y = 0$
i.e., $\bar{f} = 1$ iff $\bar{x} = \bar{y} = 1$
Hence, $\bar{f} = \bar{x} \cdot \bar{y}$, and so $f = \overline{\bar{x} \cdot \bar{y}} = x + y$

$x\ y$	f
0 0	1
0 1	0
1 0	1
1 1	1

f holds 0 iff $x = 0$ and $y = 1$
i.e., $\bar{f} = \bar{x} \cdot y$
and so $f = x + \bar{y}$

$x\ y$	f
0 0	0
0 1	0
1 0	1
1 1	1

f holds 0 iff $x = y = 0$ or $x = 0$ & $y = 1$
i.e., $\bar{f} = \bar{x} \cdot \bar{y} + \bar{x} \cdot y$
and so $f = \overline{\bar{x} \cdot \bar{y} + \bar{x} \cdot y} = (\overline{\bar{x} \cdot \bar{y}}) \cdot (\overline{\bar{x} \cdot y}) = (x + y) \cdot (x + \bar{y})$

Conjunctive Canonical Form (or POS)

SAPIENZA Università di Roma Dipartimento di Informatica

Let's have n variables $\{x_1,\ldots,x_n\}$:
A *maxterm* is a sum of n literals $l_1+\ldots+l_n$ such that

$$l_i \in \{x_i, \bar{x}_i\} \text{ for every } i \in \{1,..,n\}$$

A *conjunctive canonical form* (or POS canonical form) is a product (or a conjunction, hence the name) of pairwise distinct maxterms.

For every maxterm M, there esists one unique n-tupla of bits that make it 0.

In general, the n-tupla is obtained by assigning 0 to the variables that appear simple in M and 1 to those that appear negated.

There is a bijective correspondence between the 2^n maxterms and $\{0,1\}^n$:

$$M \leftrightarrow b_1\ldots b_n \qquad \text{iff} \qquad M(b_1\ldots b_n) = 0$$

If M is associated to $b_1\ldots b_n$ and $b_1\ldots b_n$, seen as a natural number codified in binary with n bits, corresponds to the decimal k, then M will be called M_k.

Ex.: $\bar{x}_3 + x_2 + \bar{x}_1$ holds 0 iff $x_3 = x_1 = 1$ and $x_2 = 0$
It is in bijection with 101 and so we shall call it M_5

From CCF to the BF and vice versa

SAPIENZA Università di Roma Dipartimento di Informatica

- Given a BF, the associated CCF is obtained by taking all the rows that hold 0 and by multiplying all the corresponding maxterms.
- Given a CCF, the associated BF is obtained by putting 0 in all rows whose binary strings correspond to the maxterms of the CCF and 0 elsewhere.

Ex.:

$x_3\, x_2\, x_1$	f
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	1
1 1 1	1

CCF: $M_0{\cdot}M_2{\cdot}M_4$

From a BE to its CCF

SAPIENZA Università di Roma Dipartimento di Informatica

Let E be any BE.

1. Push all negations directly on its variables (De Morgan) and delete double negations (involution)
2. Turn the expression in POS form, by using distributivity of + over ·
3. Delete possible copies of factors (idempotency) and all summands that contain a literal and its negation (complement and annihilator)

We now have a *conjunctive normal form* (or *POS*), i.e., a POS whose factors are not maxterms, in general.

4. Sum $x_i \cdot \bar{x}_i$ to every factor that doesn't contain x_i (neutral and complement)
5. Turn the resulting expression in POS form, by using distributivity of + over ·
6. Delete possible copies of factors (idempotency)

Example

SAPIENZA Università di Roma Dipartimento di Informatica

$$E = \overline{x + y\bar{z}} + \bar{y}z$$

1) $E = \bar{x}(\bar{y}+z) + \bar{y}z$

2) $= (\bar{x}(\bar{y}+z)+\bar{y})(\bar{x}(\bar{y}+z)+z) = (\bar{x}+\bar{y})(\bar{y}+z+\bar{y})(\bar{x}+z)(\bar{y}+z+z)$

3) $= (\bar{x}+\bar{y})(\bar{y}+z)(\bar{x}+z)$ → *POS Normal Form*

4) $= (\bar{x}+\bar{y}+z\bar{z})(\bar{y}+z+x\bar{x})(\bar{x}+z+y\bar{y})$

5) $= M_6 \cdot M_7 \cdot M_2 \cdot M_6 \cdot M_4 \cdot M_6$

6) $= M_7 \cdot M_6 \cdot M_4 \cdot M_2$ → *POS Canonical Form*

From a CNF to the BF

Also *conjunctive normal forms* can be used to quickly derive the truth table of a BF; the procedure id dual w.r.t. DNFs:

Every factor (sum of literals) of the CNF is associated to a set of rows in the following way:

- If x_j appears negated, x_j must hold 1;
- If x_j appears simple, x_j must hold 0;
- If x_j doesn't appear, x_j can hold 0 and 1.

Hence, we now write 0 in all rows identified in this way.

Ex.:

$$E = (\overline{x}_2 + x_1)(x_3 + x_2 + x_1)$$

x_3	x_2	x_1	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

BFs and BEs

Theorem: for every BE, there exists a unique BF associated.

Theorem: for every BF, there exists a unique BE in DCF and one unique BE in CCF associated.

Remark: unique up-to commutativity and associativity of + and · !!

By contrast, DNFs and CNFs are not unique.

Ex.:

x_3	x_2	x_1	f
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

Its DCF is $m_2 + m_3 + m_7$, i.e.

$$\overline{x}_3 x_2 \overline{x}_1 + \overline{x}_3 x_2 x_1 + x_3 x_2 x_1$$

$$= \overline{x}_3 x_2 (\overline{x}_1 + x_1) + x_3 x_2 x_1 = \overline{x}_3 x_2 + x_3 x_2 x_1$$

$$= \overline{x}_3 x_2 \overline{x}_1 + (\overline{x}_3 + x_3) x_2 x_1 = \overline{x}_3 x_2 \overline{x}_1 + x_2 x_1$$