Boolean Algebra

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Binary Codes

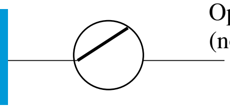
Why do we use binary codes for digital circuits?

1. *George Boole* proved that mathematical logics can be reduced to a simple algebric system, relying on a binary code (0 and 1, with 3 fundamental operations).

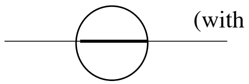
2. Binary code turned out to be fundamental also in the *commutation theory* (by Claude Shannon) to describe the behaviour of digital circuits (1=closed, 0=open).

Rélais

In Shannon's times, telephonic commutators were network of rélais (interruptors)



Open Rélais
(no signal)



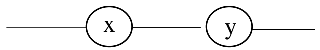
Closed Rélais
(with signal)

Shannon introduced the binary alphabeth to describe the status of the rélais

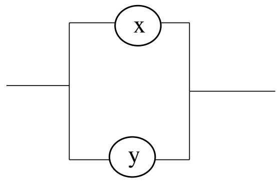
x: variable describing the status
(*x* takes values in {0,1})

Fundamental Operations On Rélais

Commutation
A closed rélais becomes open and vice versa



Serial Composition
The signal passes only if both x and y are 1, i.e. both rélais are closed



Parallel Composition
The signal passes if at least one between x and y is 1, i.e. closed

Boolean Algebra

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$$(0,1,+,\cdot,\bar{\ })$$

where:

$$+\,,\cdot\ : \{0,1\}\times\{0,1\} \rightarrow \{0,1\}$$

$$\bar{\ } : \{0,1\} \rightarrow \{0,1\}$$

The operators satisfy the following axioms:

Commutativity	$x+y = y+x$	$x\cdot y = y\cdot x$
Associativity	$x+(y+z) = (x+y)+z$	$x\cdot(y\cdot z) = (x\cdot y)\cdot z$
Distributivity	$x\cdot(y+z) = (x\cdot y)+(x\cdot z)$	$x+(y\cdot z) = (x+y)\cdot(x+z)$
Neutral element	$x+0=x$	$x\cdot 1=x$
Complement	$x+\bar{x}=1$	$x\cdot\bar{x}=0$

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Derived Laws: Idempotency

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$x\cdot x = x$

$$x = x\cdot 1 = x\cdot(x+\bar{x}) = x\cdot x + x\cdot\bar{x} = x\cdot x + 0 = x\cdot x$$

(neutral •, complement +, distrib. • +, complement •, neutral +)

$x+x = x$

$$x = x+0 = x+(x\cdot\bar{x}) = (x+x)\cdot(x+\bar{x}) = (x+x)\cdot 1 = x+x$$

(neutral +, complement •, distrib. + •, complement +, neutral •)

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Duality Principle

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From the previous example, we can notice that the proof for the law with + in place of $\cdot$ can be obtained by changing

- 0 and 1
- $+$ and $\cdot$

This always happens the Boolean Algebra and derives from the fact that all axioms satisfy the duality principle.

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Derived Laws: Annihlator Elements

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$$x\cdot 0 = 0$$

$$x\cdot 0 = x\cdot(x\cdot\bar{x}) = (x\cdot x)\cdot\bar{x} = x\cdot\bar{x} = 0$$

(complement •, associativity, idempotency, complement •)

REMARK: once proved, a derived law can be used like axioms for proving new laws

By duality, $x+1=1$

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Derived Laws: Assorbtion

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$$x + x{\cdot}y = x$$

$$x + x{\cdot}y = x{\cdot}1 + x{\cdot}y = x{\cdot}(1+y) = x{\cdot}1 = x$$

neutral • distrib. • + annihilator + neutral •

By duality, $x \cdot (x{+}y) = x$

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Derived Laws: Involution

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$$\overline{\overline{x}} = x$$

$$x = x\cdot 1 = x\cdot(\overline{x} + \overline{\overline{x}}) = x\cdot\overline{x} + x\cdot\overline{\overline{x}} =$$

Neutral · Complem. + Distrib. · +

$$= 0 + x\cdot\overline{\overline{x}} = \overline{x}\cdot\overline{\overline{x}} + x\cdot\overline{\overline{x}} = (\overline{x} + x)\cdot\overline{\overline{x}} = 1\cdot\overline{\overline{x}} = \overline{\overline{x}}$$

Complem. · Complem. · Distrib. · + Complem. + Neutral ·

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Derived Laws: De Morgan

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$$\overline{x + y} = \overline{x}\cdot\overline{y} \qquad \overline{x\cdot y} = \overline{x} + \overline{y}$$

$$\overline{x} + \overline{y} = \overline{x}\cdot(x\cdot y + \overline{x\cdot y}) + \overline{y}\cdot(x\cdot y + \overline{x\cdot y}) =$$

neutral & complement

$$= \overline{x}\cdot x\cdot y + \overline{x}\cdot\overline{x\cdot y} + \overline{y}\cdot x\cdot y + \overline{y}\cdot\overline{x\cdot y} = \overline{x}\cdot\overline{x\cdot y} + \overline{y}\cdot\overline{x\cdot y} =$$

distrib. complement, annihilator & neutral

$$= (\overline{x} + \overline{y})\cdot\overline{x\cdot y} = (\overline{x} + \overline{y})\cdot\overline{x\cdot y} + x\cdot y\cdot\overline{x\cdot y} = (\overline{x} + \overline{y} + x\cdot y)\cdot\overline{x\cdot y}$$

distrib. Neutral & complement distribut.

$$= (\overline{x} + \overline{y} + y)\cdot(\overline{x} + \overline{y} + x)\cdot\overline{x\cdot y} = (\overline{x} + 1)\cdot(\overline{y} + 1)\cdot\overline{x\cdot y} = \overline{x\cdot y}$$

distribut. complement annihilator & neutral

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From axioms to Truth Tables

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1. By Neutral element, $0 + \overline{0} = \overline{0}$
   By complement, $0 + \overline{0} = 1$
   By transitivity, $\overline{0} = 1$ and, by duality, $\overline{1} = 0$
2. By neutral element, 0+0 = 0
   By annihilator element, 0+1=1+0=1+1=1
3. By duality, 1·1 = 1 e 0·1 = 1·0 = 0·0 = 0

Hence :

x	$\overline{x}$
0	1
1	0

x y	$x{+}y$
0 0	0
0 1	1
1 0	1
1 1	1

x y	$x{\cdot}y$
0 0	0
0 1	0
1 0	0
1 1	1

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