Representing natural numbers

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Representing Information

Electronic calculators are machines that can handle *information* and convert it in other information.

In computer science, information is *whatever can be represented through proper sequences of symbols taken from a fixed (finite) alphabeth*.

A *code* C is a set of words formed by the symbols of an alphabeth Σ (called *support* of C).

Coding and decoding

CODING

A *coding function* of a set of info's I in a given code C is

$$f : \mathrm{I} \rightarrow \mathrm{C}$$

Example:

car → *00*   *shuttle* → *01*   *airplane* → *10*

where: I is a set of words

C is a subset of the words made up by 0 and 1

DECODING

A *decoding function* of a previously coded info is

$$g : \mathrm{C} \rightarrow \mathrm{I}$$

(typically, it is the inverse function of *f*)

Example:

00 → *macchina*   *01* → *razzo*   *10* → *aereo*

Valuation Criteria for a coding

Economicity: codes that use less symbols are considered better

Simplicity of coding/decoding: practical performances are desirable

Simplicity of elaboration: we look for codes that provide easier ways of executing operations (e.g., roman numeric system does not provide an easy way of performing sums/subtractions – the mechanisms of carry/borrow fail).

Positional Systems

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A *positional numeric system* in base b, i.e. relying on an **alphabeth** Σ made up of b distinct symbols, allows us to express any natural number N of m digits, through the formula

$$N = \sum_{i=0}^{m-1} c_i b^i, \quad c_i \in \Sigma$$

For example, in the decimal system (b=10, Σ=0,1,..9), the sequence $N_{10} = 254$ expresses the number

$$254_{10} = 200 + 50 + 4$$
$$= 2\cdot10^2 + 5\cdot10^1 + 4\cdot10^0$$

Similarly, in base 7 (b=7, Σ=0,1,..6), the same sequence $N_7 = 254$ expresses the number

$$254_7 = 2\cdot7^2 + 5\cdot7^1 + 4\cdot7^0 \; (= 98+35+4 = 137)_{10}$$

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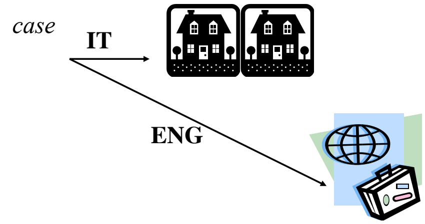
Hence, 10_{10} and 10_2 have a **different** meaning, even if they are the same sequence of symbols (1 followed by 0)!!

To understand the meaning of a sequence, we must know the *decoding function* to use

Example:



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	Base 2	Base 3	Base 4	Base 5	…	Base 8	…	Base 10	…	Base 16
	0000	000	00	00		00		00		0
	0001	001	01	01		01		01		1
	0010	002	02	02		02		02		2
	0011	010	03	03		03		03		3
	0100	011	10	04		04		04		4
	0101	012	11	10		05		05		5
	0110	020	12	11		06		06		6
N_b	0111	021	13	12		07		07		7
	1000	022	20	13		10		08		8
	1001	100	21	14		11		09		9
	1010	101	22	20		12		10		A
	1011	102	23	21		13		11		B
	1100	110	30	22		14		12		C
	1101	111	31	23		15		13		D
	1110	112	32	24		16		14		E
	1111	120	33	30		17		15		F

Binary Code

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Binary code: a code made up just from the symbols 0 and 1.

So, b=2 e Σ ={0,1}

0 ed 1 are called *bit*, a contraction for *binary digit*.

Why binary code?

- *George Boole* proved that al logic can be reduced to a simple algebraic system using the binary code.
- Binary code turned out very useful in the *commutation theory* (by Claude Shannon) to describe the behavious of digital circuits (1=on, 0=off).

Every info can be codified in binary

- Let's start with numbers: codes will be **different** according to what kind of numbers we have to represent (naturals, integers, rationals, …)
- We can also encode words (sequences of alphabetical characters, through ASCII and/or UNICODE codes), images, sounds, …

Binary Representation of Naturals

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A **natural number** N in base 2 with n bits is represented by the formula:

$$N_2 = \sum_{i=0}^{n-1} c_i 2^i,\ c_i \in \{0,1\}$$

Bit c_0 is called LSB (*less signifying bit*) whereas c_{n-1} is called MSB (*most signifying bit*).

Example: $111001_2 = 1\cdot2^5+1\cdot2^4+1\cdot2^3+0\cdot2^2+0\cdot2^1+1\cdot2^0$

Changing the base

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Problem: given a number N in base a (N_a),

turn it into a number N' in base b (N'_b)

Base conversion: polynomial method

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We use the formula

$$N_a = \sum_{i=0}^{n-1} c_i a^i,\ c_i \in \{0,\dots,a-1\}$$

That is, we express N_a as a polynomium, by **using digits of the alphabeth b** in the polynomium

Then, we evaluate the polynomium by **using arithmetics in base b**

Example (from base 2 to base 10):

$111001_2 = (1\cdot2^5+1\cdot2^4+1\cdot2^3+0\cdot2^2+0\cdot2^1+1\cdot2^0)_{10}$

$= (32+16+8+0+0+1)_{10} = 57_{10}$

Polynomial Method

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Hard to use if the final base (b) is not 10
(you have to work with the arithmetics modulo b!!)

Example (from base 7 to base 3):
$3602_7 = (10\cdot21^3 + 20\cdot21^2 + 0\cdot21^1 + 2\cdot21^0)_3$

WE NEED ANOTHER METHOD!!!

Theorem of the Euclidean Division

For every $D,d \in \mathbf{N}$, there exists a unique pair $q,r \in \mathbf{N}$ s.t.

$$D = q \cdot d + r \,, \quad \text{with } 0 \leq r < d$$

dividend, quotient, divisor, reminder

Notationally, $q = D \text{ div } d$
$r = D \text{ mod } d$

For example, $7 \text{ div } 3 = 2$
$7 \text{ mod } 3 = 1$

$$N = c_0 + c_1 \cdot b^1 + c_2 \cdot b^2 + c_3 \cdot b^3 + \ldots + c_{n-2} \cdot b^{n-2} + c_{n-1} \cdot b^{n-1}$$

$$= c_0 + b \cdot \underbrace{(c_1 + c_2 \cdot b^1 + c_3 \cdot b^2 + \ldots + c_{n-2} \cdot b^{n-3} + c_{n-1} \cdot b^{n-2})}_{\text{quotient}}$$

dividend, reminder, divisor

From which

$N \text{ mod } b = c_0$

$N \text{ div } b = c_1 + c_2 \cdot b^1 + c_3 \cdot b^2 + \ldots + c_{n-2} \cdot b^{n-3} + c_{n-1} \cdot b^{n-2} = N^{(1)}$

By iterating the reasoning

$N^{(1)} \text{ mod } b = c_1$

$N^{(1)} \text{ div } b = c_2 + c_3 \cdot b^1 + \ldots + c_{n-2} \cdot b^{n-4} + c_{n-1} \cdot b^{n-3} = N^{(2)}$

$N^{(2)} \text{ mod } b = c_2$

$N^{(2)} \text{ div } b = c_3 + \ldots + c_{n-2} \cdot b^{n-5} + c_{n-1} \cdot b^{n-4} = N^{(3)}$

…

Until we arrive at $N^{(n-1)} = c_{n-1}$ and then we have

$N^{(n-1)} \text{ mod } b = c_{n-1}$

$N^{(n-1)} \text{ div } b = 0$

Base conversion: iterated divisions method

To convert N_a in base b

1. Repeatedly divide N_a for b_a
   (IMP.: **b must be expressed in base a and the division is done in base a**)

2. The remainders of the divisions, converted in base b, give us the digits, from the less to the most signifying one, of N_a expressed in base b.

Example: 657_{10} in base 4

657 : 4 = 164 reminder 1
164 : 4 = 41 reminder 0
41 : 4 = 10 reminder 1
10 : 4 = 2 reminder 2
2 : 4 = 0 reminder 2

Hence, $657_{10} = 22101_4$

Example (from base 10 to base 16)

We have to perform the division of 317_{10} by 16_{10}
(in base 16, digits are 0,1,…,9,A,B,…,F)

1) 317 **:** 16
$Q=19_{10}$, $r=13_{10}$
$13_{10}=\mathbf{D}_{16}$ (LSD)

2) 19 **:** 16
$Q=1_{10}$, $r=3_{10}$
$3_{10}=\mathbf{3}_{16}$

3) 1 **:** 16
$Q=0_{10}$, $r=1_{10}$
$1_{10}=\mathbf{1}_{16}$ (MSD)

Hence,

$317_{10}=\mathbf{13D}_{16}$

N.B.: in the previous algorithm, the division **is in base *a* (i.e., in the base of the starting number)**. If $a \neq 10$, this is difficult!

Example (starting and arrival bases are not 10)

Convert 102202_3 in base 5

Three ways:

a) Repeatedly calculate 102202_3 **:** 12_3

division with arithmetics in base 3 → **DIFFICULT!**

b) Use the polynomial method for 102202_3

products and sums with arithmetics in base 5 → **DIFFICULT!**

c) Convert 102202_3 in base 10 (polynomial meth.) and then convert the result in base 5 (iterated divisions)

- $102202_3 = 3^5 + 2 \cdot 3^3 + 2 \cdot 3^2 + 2 = 317_{10}$
- 317:5 = 63 reminder 2
  63:5 = 12 reminder 3
  12:5 = 2 reminder 2
  2:5 = 0 reminder 2

Hence, $102202_3 = 2232_5$

Conversions from base *a* to base a^k

Prop.: in the arithmetics in base *a* we have that

$$c_{n-1} \ldots c_1 c_0 \ \text{mod}\ a^k = c_{k-1} \ldots c_0$$
$$c_{n-1} \ldots c_1 c_0 \ \text{div}\ a^k = c_{n-1} \ldots c_k$$

Example ($a = 10$ and $k = 2$): 453_{10} mod $100 = 53$
453_{10} div $100 = 4$ (since $100 = 10^2$)

So, if $b = a^k$ and $N_a = c_{n-1} \ldots c_1 c_0$, then the number in base b is

$$(c_{n-1} \ldots c_{hk})_b \ldots (c_{3k-1} \ldots c_{2k})_b (c_{2k-1} \ldots c_k)_b (c_{k-1} \ldots c_0)_b$$

Specific case: *Conversions from base 2 to base* 2^k

Take the bits in k-tuples starting from the LSB and convert them in base 2^k

Example: convert 1000111101 from base 2 to base 4 ($= 2^2$)

$(10\ 00\ 11\ 11\ 01)_2 = (2\ 0\ 3\ 3\ 1)_4$

Example

Convert 101001101101 from base 2 to base 8 and 16

OBS: $8=2^3$ and $16=2^4$

First conversion: gruop in triples and convert:

$(101\ 001\ 101\ 101)_2 \rightarrow (5\ 1\ 5\ 5)_8$

Second conversion: group in quadruples and convert

$(1010\ \ 0110\ \ 1101)_2 \rightarrow (A\ 6\ D)_{16}$

OBS: in making the k bits groups, the most signifying group can have less than k bits (if the length of the starting sequence is not a multiple of k); in this case, we add 0's on top

Ex.: 1001101101 from base 2 to base 8: 001 001 101 101

Conversions from base b^k to base *b*

Similarly, if $a = b^k$ and $N_a = c_{n-1} \ldots c_1 c_0$, then the number in base b is

$$(c_{n-1})_b \ldots (c_2)_b (c_1)_b (c_0)_b$$

where every c_i is converted in base b by using k digits.

Example 1: Convert 8315_9 in base 3

Since $9 = 3^2$, we can convert every digit of the given number in base 9 by using two digits in base 3:

$8\ 3\ 1\ 5_9 = 22\ 10\ 01\ 12_3$

Example 2: Convert $8D3A_{16}$ in base 2

Since $16 = 2^4$, we can convert every digit of the given number in base 16 with 4 bits:

$8\ D\ 3\ A_{16} = 1000\ 1101\ 0011\ 1010_2$

To convince yourselves of the correctness of this method, do the conversions in base 10, as explained in the previous slides!

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Base Conversions: Summing up

- The polynomial method is easy if the arrival base is 10
- Iterated divisions method is easy if the starting base is 10
- If neither the starting nor the arrival base is 10:

General solution:
- Convert N_a in base 10 with the polynomial method
- Convert the obtained result in base b with the iterated divisions method

If the arrival base is a power of the starting base ($b = a^k$):
- Convert in base b k-tuples of digits, from the less to the most signifying, of N_a
- The digits obtained in this way give the digits, from the less to the most signifying one, of the number represented in base b

If the starting base is a power of the arrival base ($a = b^k$):
- Convert in base b every digit of N_a by using k digits (of base b)

Number of binary sequences

If we have n bits, how many different binary sequences can we obtain?

1 bit:	0 ,	1	→ 2 sequences
2 bits:	00,01 ,	10,11	→ 4 sequences
3 bits:	000,001,010,011,	100,101,110,111	→ 8 sequences
4 bits:	…	…	→ 16 sequences

…

At every step, we double the sequences of the previous step:

n bits: $\underbrace{2 \cdot 2 \cdot 2 \ldots \cdot 2}_{n} = 2^n$ sequences

Representation Interval

Hence, with n bits, we can represent 2^n numbers: $\{0,\ldots,2^n-1\}$

Indeed, the smallest number is

$$\underbrace{0 \ldots 0}_{n} = \underbrace{0 + \ldots + 0}_{n} = 0$$

whereas the biggest number is

$$\underbrace{1 \ldots 1}_{n} = \sum_{i=0}^{n-1} 2^i = 2^n - 1$$ (this is the geometrical series!)

Representation Length

How many bits do we need to represent N_2?

We have to find the smallest $n \in \mathbf{N}$ such that

$$N < 2^n$$

OBS: $\log_b k$ is the exponent that must be given to the base (b) to obtain the argument (k), that is

$$k = b^{\log_b k}$$

Hence, by letting $k = N$ and $b = 2$, the solution of $N = 2^n$ is $n = \log_2 N$

OBS.: in general, n is an irrational number. Since we need a natural (and we don't need the exact equality), we take $\lfloor \log_2 N \rfloor + 1$

Examples: $N = 57$: $\log_2 N = 5{,}8328\ldots$

hence, I need 5+1 bits (indeed, $57_{10} = 111001_2$)

$N = 64$: $\log_2 N = 6$

hence, I need 6+1 bits (indeed, $64_{10} = 1000000_2$)

Codewords length

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For simplicity, computers work on words of fixed length, that are typically powers of 2:

- 8 bits = *byte*
- 16 bits = *half-word*
- 32 bits = *word*
- 64 bits = *long word*

Let k be the adopted codeword length:

- If a number can be represented with exactly k bits, we're OK
- If a number can be represented with less than k bits (say m), we have to add k-m 0's in the most signifying positions
- If a number needs more than k bits to be represented?
  → error situation, called *overflow*