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Optimal three-dimensional layout of interconnection networks

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Abstract

The main benefits of a three-dimensional layout of interconnection networks are the savings in material (measured as volume) and the shortening of wires. The result presented in this paper is a general formula for calculating a lower bound on the volume. Moreover, for butterfly and X-tree networks we show layouts optimizing the maximum wire length and whose upper bounds on the volume are close to the lower bounds. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Three-dimensional layout; Butterfly network; X-tree network; Minimum bisection width

1. Introduction and preliminaries

The importance of representing interconnection networks in three-dimensions and the benefits derived from efficient designs have already been known since the 1980s by Rosenberg [19], and then in [18, 13]. However, only recently, have hardware advances allowed three-dimensional circuits to have a cost low enough to make them commonly available. For this reason three-dimensional layouts of graphs on rectilinear grids are becoming of wide interest both in the study of the VLSI layout problem for integrated circuits and in the study of algorithms for drawing graphs. Indeed, the tie between VLSI layout studies and theoretical graph drawing is very strong since to lay out a network on a grid is equivalent to drawing orthogonally the underlying graph.

In the past few years, the research on three-dimensional graph drawing has been prolific [2, 3, 5, 6, 8–10, 15, 16]. However, the papers dealing with orthogonal three-

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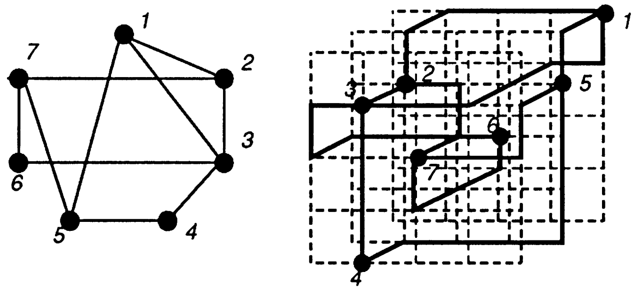


Fig. 1. Example of a graph and its three-dimensional layout.

dimensional drawing of graphs with bounded degree 6, e.g. [2, 6, 9, 10], show results that are valid for very general graphs. In particular, most of them give algorithms drawing a graph $G(V,E)$ in $O(|V|^{3/2})$ volume and with very few bends per edge. These results are not satisfactory for structured and regular graphs such as interconnection network topologies. Furthermore, the most relevant aims for the layout problem are to shorten wires and to save material (measured as volume). Indeed, shortening wires sets the target of improving the communication time and this is a fundamental requirement for interconnection networks, together with the minimization of the layout volume. The number of bends, on the other hand, is a parameter of minor relevance.

In this paper we focus our attention on the orthogonal three-dimensional grid layout of interconnection networks which is based on the following formal definitions (Fig. 1).

Definition 1. The $h \times w \times l$ *three-dimensional grid* is the graph whose node set is the set of triples (i,j,k), $1 \leqslant i \leqslant h$, $1 \leqslant j \leqslant w$, $1 \leqslant k \leqslant l$ and whose edges connect nodes (a,b,c) and(d,e,f) just when $|a-d| + |b-e| + |c-f| = 1$.

In the following, we will call *x-lines* (respectively, *y-lines*, *z-lines*) the subgraphs of a three-dimensional grid induced by nodes (i,j,k) such that j and k (respectively, i and k, i and j) are fixed and i varies in the range $[1,w]$ (respectively, j varies in $[1,l]$ and k in $[1,h]$).

Definition 2. A *three-dimensional grid layout* of a graph $\mathscr{G}$ is a mapping of $\mathscr{G}$ in the three-dimensional grid such that nodes are mapped to grid-nodes and edges are mapped to independent paths satisfying the following conditions:
- distinct paths are edge-disjoint (then at most three paths can cross at a grid-node);
- 'knock-knee' paths [14] are not allowed, i.e. if a path presents a bend into a grid-node only another path can pass through the same grid-node, and it must cross the grid-node in straight-line fashion;
- a path may not touch a mapped node, except at its endpoints.

If the layout of a graph $\mathscr{G}$ can be enclosed in a $h \times w \times l$ three-dimensional grid, we say the *layout volume* of $\mathscr{G}$ is the product of $h \times w \times l$.

In this work we study the three-dimensional layout of interconnection networks. The contribution of this paper is twofold. In the first part we prove a general formula for calculating a lower bound on the three-dimensional layout volume. This formula puts the layout volume into relation with parameters peculiar of the network, i.e. MSBW and congestion. It improves the result in [19] since it provides a precise value instead of an order of magnitude. In the second part we focus on methods to lay out some networks. In particular, we first introduce the new notion of k-3D double channel routing and we use it to exhibit an optimal three-dimensional drawing for butterfly. Beyond this, we show an optimal layout for X-tree networks. The achieved volume values are $O(N^{1/2}) \times O(N^{1/2}) \times O(N^{1/2})$ and $O(N^{1/3}) \times O(N^{1/3}) \times O(N^{1/3})$ for butterfly and X-tree networks, respectively, and both are optimal because their values match their respective lower bounds. Concerning the wire length, the method to lay out the butterfly allows all wires to be $O(N^{1/4})$ long, and only one edge level has $O(N^{1/2})$ wire length. This result is an improvement on the previously known layouts [13, 18, 19] — in which all wires were $O(N^{1/2})$ long — in view of the observations made in Section 3.1. The maximum wire length in our layout of a N leaf X-tree is $O(N^{1/3})$. To the best of our knowledge no previous results are known about X-tree. Observe that if we run any three-dimensional graph drawing algorithm we would have achieved much worse bounds, since our results are a function of the number of input nodes (butterfly) or of leaves (X-tree), that are a proper subset of the set of nodes, and not — as in the case of algorithms for general graphs — a function of the number of nodes.

This paper is organized as follows: in Section 2 we give some preliminary definitions and we prove our lower bound formula. In Section 3 we show the method to optimally lay out butterfly and X-tree networks. Finally, in Section 4, we address some open problems about laying out some other interconnection networks.

2. Lower bound

In this section we prove a general formula giving a lower bound on the layout volume of interconnection networks. Readers not interested in the details of this proof can omit this section without compromising the comprehension of next section in which methods to lay out butterfly and X-tree networks are described.

We obtain our result by generalizing to three dimensions the classical lower bound strategy for two dimensions invented in [20], and modified in [1]. Before proving the general formula for the lower bound, we give some definitions and prove some preliminary results.

Definition 3. An *embedding ε of graph $\mathscr{G}$ into graph $\mathscr{H}$* (which has at least as many nodes as $\mathscr{G}$) comprises a *one-to-one association* α of the nodes of $\mathscr{G}$ with nodes of $\mathscr{H}$, plus a routing ρ which associates each edge $\{u, v\}$ of $\mathscr{G}$ with a path in $\mathscr{H}$ that connects nodes $\alpha(u)$ and $\alpha(v)$.

The *congestion* of embedding ε is the maximum, over all edges e in $\mathscr{H}$, of the number of edges in $\mathscr{G}$ whose ρ-routing paths contain edge e.

Definition 4. Let $\mathscr{G}$ be a graph having a designated set of $2c>0$ nodes, called *special nodes*. The *minimum special bisection width* of a graph $\mathscr{G}$, $MSBW(\mathscr{G})$, is the smallest number of edges whose removal partitions $\mathscr{G}$ into two disjoint subgraphs, each containing half of $\mathscr{G}$'s special nodes.

Lemma 5 (Leighton [12]). *Let ε be an embedding of graph $\mathscr{G}$ into graph $\mathscr{H}$ that has congestion C, then the following inequality holds*:

$$MSBW(\mathscr{H}) \geqslant \frac{1}{C} MSBW(\mathscr{G}).$$

Now we prove a general formula to get a lower bound on the layout volume of a network, given its MSBW.

Lemma 6. *For any graph $\mathscr{G}$, the volume of the smallest three-dimensional layout of $\mathscr{G}$ is at least $(\sqrt{MSBW(\mathscr{G})}-1)^3$.*

Proof. We consider an arbitrary layout of $\mathscr{H}$ in the grid of dimension $h \times w \times l$, where, without loss of generality, $w, l \leqslant h$. Let $P_0=(x_0, y_0, z_0)$ be a grid-node. We define a surface S with respect to P_0 as a surface whose x-coordinates are:

$$x = x_0 + \tfrac{1}{2} \quad \text{if either } (y \leqslant y_0 + \tfrac{1}{2} \text{ and } z < z_0 - \tfrac{1}{2}) \text{ or } (y > y_0 + \tfrac{1}{2} \text{ and } z < z_0 + \tfrac{1}{2})$$

$$x = x_0 - \tfrac{1}{2} \quad \text{if either } (y < y_0 + \tfrac{1}{2} \text{ and } z > z_0 - \tfrac{1}{2}) \text{ or } (y \geqslant y_0 + \tfrac{1}{2} \text{ and } z > z_0 + \tfrac{1}{2})$$

$$x_0 - \tfrac{1}{2} \leqslant x \leqslant x_0 + \tfrac{1}{2} \quad \text{if either } (y < y_0 + \tfrac{1}{2} \text{ and } z = z_0 - \tfrac{1}{2}) \text{ or } (y > y_0 + \tfrac{1}{2} \text{ and } z = z_0 + \tfrac{1}{2}) \text{ or } (y = y_0 + \tfrac{1}{2} \text{ and } z_0 - \tfrac{1}{2} \leqslant z \leqslant z_0 + \tfrac{1}{2})$$

Informally speaking, S can be described in the following way (see Fig. 2):

- S has an "indentation" J located by a line that has a single unit-length "step" J' such that the line is aligned with the y-lines in such a way that the portion of J above "step" J' lies above some P_0; "step" J' lies to the right side of some P_0; the portion of J under "step" J' either lies outside of the grid or it lies under P_0;
- S is orthogonal to the x-line of the grid in such a way that the portion of S above "indentation" J lies behind plane π with equation $x=x_0$; the portion of S below "indentation" J either lies outside of the grid, or it lies in front of plane π.

It is not hard to prove that S can be positioned on the grid in such a way that it cuts the layout of $\mathscr{H}$ into two subgraphs, each containing half of $\mathscr{H}$'s special nodes.

Removing the grid-edges crossed by S yields a bisection of $\mathscr{H}$. By definition, at least $MSBW(\mathscr{H})$ edges of $\mathscr{H}$ must cross surface S. By construction, at most $hl+l+1$

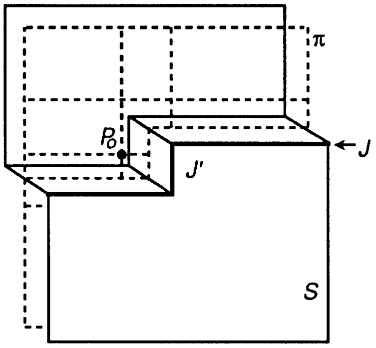


Fig. 2. Surface S with the jog J.

edges of the grid cross surface S. It follows that $hl + l + 1 \leqslant h^2 + h + 1$ and $hl + l + 1 \geqslant MSBW(\mathscr{H})$. Since $h^2 + h + 1$ is less than $(h+1)^2$, we have $h \geqslant \sqrt{MSBW(\mathscr{H})} - 1$, hence the lemma follows. □

As a consequence of Lemmas 6 and 5, if $MSBW(\mathscr{H})$ is not known, a lower bound on the layout volume of a network $\mathscr{H}$ can be computed through an embedding ε into $\mathscr{H}$ of a graph $\mathscr{G}$ if $MSBW(\mathscr{G})$ and the congestion C of ε are known. In this way, we have that:

$$\text{lower bound on the layout volume of } \mathscr{H} \geqslant (\sqrt{MSBW(\mathscr{H})} - 1)^3 \geqslant \left(\sqrt{\frac{1}{C} MSBW(\mathscr{G})} - 1\right)^3.$$

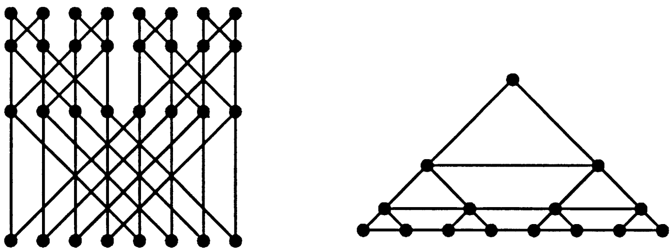
Since another lower bound on the layout volume of a graph is trivially given by the number of nodes of the graph, from the previous considerations the following theorem derives:

Theorem 7. *Given a graph $\mathscr{H}$ with n nodes, a lower bound on its layout volume is given by* $\max\{n, (\sqrt{MSBW(\mathscr{H})} - 1)^3\}$. *Alternatively, when an embedding of congestion C for an auxiliary graph $\mathscr{G}$ into $\mathscr{H}$ and $MSBW(\mathscr{G})$ are known, a lower bound on the layout volume of $\mathscr{G}$ is* $\max\{n, (\sqrt{\frac{1}{C} MSBW(\mathscr{G})} - 1)^3\}$.

3. Upper bound of some interconnection networks

In this section we first give the definitions of the networks we are going to manage, then we show a method to lay out both of them in a three-dimensional grid.

Definition 8. The *butterfly network* having N inputs $\mathscr{B}_N$, where $N = 2^n$, has nodes corresponding to pairs $\langle w, l \rangle$ where l is the level $(1 \leqslant l \leqslant \log N + 1)$ and w is a $\log N$-



a b

Fig. 3. a. $\mathscr{B}_8$; b. $\mathscr{T}_8$.

bit binary number that denotes the column of the node. Two nodes $\langle w, l\rangle$ and $\langle w', l'\rangle$ are linked by an edge if and only if $l' = l + 1$ and either:
1. w and w' are identical (straight-edge), or
2. w and w' differ in precisely the l'th bit (cross-edge).

Lemma 9 (Even and Litman [11]). *The subgraph of $\mathscr{B}_N$ induced by the nodes of levels $1,\ldots,h$ is the disjoint sum of $2^{\log N-h+1}$ copies of $\mathscr{B}_{2^{h-1}}$ and the subgraph of $\mathscr{B}_N$ induced by the nodes of levels $h,\ldots,\log N + 1$ is the disjoint sum of 2^{h-1} copies of $\mathscr{B}_{2^{\log N-h+1}}$.*

Definition 10. The N-leaf *X-tree* $\mathscr{T}_N$, where $N = 2^n$, is a complete N-leaf binary tree with edges added to connect consecutive nodes on the same level of the tree.

In Fig. 3 a $\mathscr{B}_8$ and a $\mathscr{T}_8$ are shown.

We can utilize Theorem 7 to compute, in particular, a lower bound on the layout volume of the interconnection networks just defined

- A lower bound on the layout volume of a butterfly network $\mathscr{B}_N$ can be obtained by considering the embedding described in [1]. The guest graph is the complete bipartite graph $K_{N,N}$, whose $MSBW$ is $N^2/2$ and the congestion of such an embedding is $N/2$. From Theorem 7 a lower bound for $\mathscr{B}_N$ is $((N-1)/2)^{3/2}$.
- The number of nodes of an N-leaf X-tree $\mathscr{T}_N$, constitutes a lower bound on its layout volume, that is $2N-1$. Indeed, the formula involving $MSBW(\mathscr{T}_N) = \log N+1$, produces a worse value. It is possible to slightly improve this lower bound to $3N-2$, by observing each triangle in the graph cannot be lain out without bends and that the number of edge-disjoint triangles in an N-leaf X-tree is exactly $N-1$.

For what concerns the upper bound on the layout volume of butterfly and X-tree networks, we divide the next part into two subsections, one for each network.

3.1. Butterfly network

It is easy to obtain an optimal three-dimensional layout of a N input butterfly network by using the forerunner intuition of Wise [21] to better visualize a butterfly network in the space. This idea is based on opportunely putting and connecting in the space $O(\sqrt{N})$ copies of any bidimensional optimal drawing of a butterfly with $O(\sqrt{N})$ inputs (possible in view of Lemma 9). The necessary volume for this layout is $O(N^{3/2})$, that is the same order of magnitude of the lower bound. A drawback of such a drawing is that the maximum wire length is $O(\sqrt{N})$, and most of the wires reach this upper bound. Also in [18] and [19] two methods for laying out a butterfly network in $O(N^{3/2})$ volume and $O(N^{1/2})$ maximum wire length are presented. Further, in [13], a method to transform a two-dimensional layout into a three-dimensional one is explained. Unfortunately all these works describe layouts whose maximum wire length is $O(N^{1/2})$ and almost all the wire reach this bound.

In the following, we will describe a method to lay out a butterfly network with N inputs in the three-dimensional grid such that almost all its wires have maximum length $O(N^{1/4})$ and only one (additive) edge-level is characterized by having maximum wire length $O(\sqrt{N})$. Observe that the wires $O(\sqrt{N})$ long all belonging to the same level imply that any data flux from input to output nodes has only one step whose time is proportional to $O(\sqrt{N})$, among $\log N$ steps whose times are proportional to $N^{1/4}$.

From now on, we will assume that $\log N$ is even; when $\log N$ is odd it is easy to adjust the details, so we omit that case for sake of brevity.

In view of Lemma 9 we can 'cut' $\mathscr{B}_N$ along its median node-level and get $\sqrt{N}$ copies of $\mathscr{B}_{\sqrt{N}}$ (O-group) whose output nodes must be re-connected to the input nodes of other $\sqrt{N}$ copies of $\mathscr{B}_{\sqrt{N}}$ (I-group) through an additive edge-level.

Hence, our layout consists of two main steps:
- three-dimensional layout of each copy of $\mathscr{B}_{\sqrt{N}}$;
- re-connection of the two groups of $\sqrt{N}$ copies of $\mathscr{B}_{\sqrt{N}}$ through an additive edge-level.

3.1.1. *Three-dimensional layout of each copy of $\mathscr{B}_{\sqrt{N}}$*

In order to accomplish this step, we exploit the following observation:

Observation 1. *An N-input butterfly network $\mathscr{B}_N$ can be covered by N edge-disjoint complete binary trees as follows*:
- *for any $i=3,\dots,\log N$, there are $2^{\log N-i}$ tree T_i having i node-levels, sharing their leaves with some tree T_j, $j>i$, and their internal nodes with some T_k, $k<i$;*
- *there are $N/2$ trees T_2 having 2 node-levels, sharing their leaves with some T_j, $j>2$;*
- *there are two trees $T_{\log N+1}$ having $(\log N+1)$ node-levels, sharing their leaves each other, and their internal nodes with some T_k, $k<\log N+1$.*

An example of this covering for $\mathscr{B}_{16}$ is depicted in Fig. 4.

Consider an H-tree representation of $T_{\log\sqrt{N}+1}$, call it $H_{\log\sqrt{N}+1}$. Call H_i, $i=2,\dots,\log\sqrt{N}$, a plane representation of T_i obtained from $T_{\log\sqrt{N}+1}$ by eliminating superfluous $\log\sqrt{N}+1-i$ levels. Then T_i is represented according to an H-tree scheme wasting some area. Observe that if the leaves of a tree T_j coincide with some internal nodes

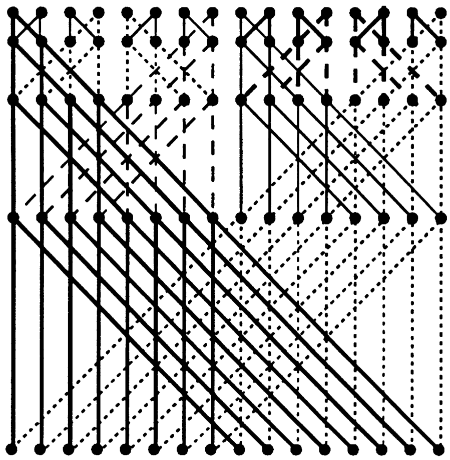


Fig. 4. Tree-covering of $\mathscr{B}_{16}$ (different trees are represented by different line types).

of a tree T_i, $i>j$, it is possible to draw T_i and T_j in the three-dimensional grid by considering H_i and H_j on two parallel planes, such that the orthogonal projection of H_j on the plane containing H_i coincides, level by level, with H_i itself. To correctly connect H_i and H_j we have to connect duplicate nodes by a segment orthogonal to both planes and to eliminate the leaves of H_j, substituting them with bends (see Fig. 5).

Based on Observation 1, we need to detail in which order the planes containing the $\sqrt{N}$ binary trees must be arranged. The following recursive pseudo-code allows one to assign a z-coordinate to each plane containing T_j ($z \leftarrow T_j$ for short). The first call of the procedure is PUT($T_{\log\sqrt{N}+1}$, 0).

```
PROCEDURE PUT(T_j, VAR z);
BEGIN
  z ← T_j;
  z + 1 ← (T_2 sharing its leaves with level 2 of T_j):
  i := 3;
  WHILE (i < j) DO
  BEGIN
    PUT (T_i sharing its leaves with level i of T_j, z + 2);
    i := i + 1;
  END;
END.
```

The order of the planes located by the previous procedure is such that the trees are drawn from the smaller one to the larger ones, and then again recursively. At the end of the procedure, half of $B_{\sqrt{N}}$ has been laid out. The remaining part can be symmetrically drawn in such a way that the planes containing the trees $T_{\log\sqrt{N}+1}$ are consecutive.

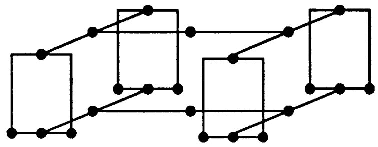


Fig. 5. Drawing of two trees sharing some nodes.

As far as the procedure is concerned, vertical lines are guaranteed:

- not to cross trees-nodes of intermediate planes; indeed, the procedure draws the trees connected to a certain tree T_j such that the smaller they are the closer to T_j they are positioned;
- not to coincide with other vertical lines; indeed, no more than two trees share the same nodes.

Based on the construction of the three-dimensional layout of $\mathscr{B}_{\sqrt{N}}$, on Observation 1 and on the area of an H-tree, it follows that each butterfly $\mathscr{B}_{\sqrt{N}}$ belonging both to the O-group and to the I-group take a $(2N^{1/4}-1)\times(2N^{1/4}-1)\times(N^{1/2})$ volume.

Observe that this layout is not suitable to draw a $\mathscr{B}_N$ in optimal volume, since such a method would lead to a $\mathrm{O}(N^2)$ volume for a $\mathscr{B}_N$.

3.1.2. Re-connection between the two groups of $\sqrt{N}$ copies of $\mathscr{B}_{\sqrt{N}}$

Let us consider the two groups of $\sqrt{N}$ copies of $\mathscr{B}_{\sqrt{N}}$. Each group is positioned in the space to form a square with $N^{1/4}$ copies on each side, such that the correspondent trees of each copy lie on the same plane. The two groups are then positioned one in front of the other. Now we have to connect the duplicated nodes through an additive edge-level.

Before detailing this operation, we need to recall some known notions and to introduce some new results.

Definition 11. A *k-channel routing* involves a bidimensional grid and two sets S and S' each consisting of k nodes to be connected by a 1–1 function. S and S' are arranged on two opposite sides of the grid.

Pinter gives a bound on the size of the considered grid.

Lemma 12 (Leighton [17]). *The grid involved in any k-channel routing is not greater than $(k+1)\times(\frac{3}{2}k+2)$ and S and S' lie on the shorter sides.*

Coming back to the butterfly problem, observe that all the output nodes of the O-group and all the input nodes of the I-group can be provided by an outgoing link towards the opposite group and their extremes can be led to two parallel planes, having empty intersection with the layouts of each copy. If we number in the same way — from left to right, row by row — both the output nodes of any butterfly of the O-group and the input nodes of any butterfly of the I-group and the butterflies themselves of

O- and I-groups, then each edge must connect the ith output node of the jth butterfly in the O-group to the jth input node of the ith butterfly in the I-group. Furthermore, it is easy to see that each row of the output nodes in the O-group is routed to a row of input nodes in the I-group.

In order to solve this problem we define a new three-dimensional constrained routing, called k-3D double channel routing, to which we reduce the previous problem.

Definition 13. A k-3D *double channel routing* involves a three-dimensional grid (the channel) and two sets S and S', both of k nodes, to be connected by a 1–1 function f. S and S' are arranged on two opposite sides of the three-dimensional grid, on the nodes of a $\sqrt{k} \times \sqrt{k}$ grid. Function f associates to a node (x, y) of S a node (x', y') of S' such that $x' = g(x)$ and $y' = h(y)$, where functions g and h are two-dimensional $\sqrt{k}$-channel routings.

We call this special case of channel routing *double* since it can be seen as a channel routing between rows of S and rows of S', and a channel routing between nodes of a fixed row of S and nodes of the corresponding row in S'.

Related to this new definition, we can give the following result.

Theorem 14. *A three-dimensional grid of size* $(\sqrt{k}+1) \times (\sqrt{k}+1) \times (\frac{3}{2}\sqrt{k}+2)$ *is enough to realize a* k*-3D double channel routing.*

Proof. Project the three-dimensional grid of the k-3D double channel routing on plane xz. It is easy to see that function g mapping rows of S in rows of S' can be considered as a two-dimensional channel routing on plane xz. Therefore, a $(\sqrt{k}+1) \times (\frac{3}{2}\sqrt{k}+2)$ two-dimensional grid is enough to realize such a channel routing (Lemma 12). When coming back to three dimensions, lines drawn to represent function g become (bent) planes. Each of such planes has on opposite horizontal sides a row x of S and its corresponding row $g(x)$ of S' and it is at least $\frac{3}{2}\sqrt{k}+2$ long (see Fig. 6). Therefore, on each plane we can realize a two-dimensional channel routing given by function h, simply by adding an extra-plane, parallel to plane xz. □

It is straightforward to use this theorem to lay out the additive edge-level between the O-group and the I-group in at most $\frac{3}{2}\sqrt{N}+2$ height.

Recombining all the arguments about the volume needed by the two operations of laying out each copy of $\mathscr{B}_{\sqrt{N}}$ and re-connecting the two groups of $\sqrt{N}$ copies of $\mathscr{B}_{\sqrt{N}}$, we obtain a $(2N^{1/2} - N^{1/4} + 1) \times (2N^{1/2} - N^{1/4} + 1) \times (\frac{7}{2}N^{1/2} + 2)$ volume for the three-dimensional layout of a $\mathscr{B}_N$, $\log N$ even.

For the general case, we can state the following theorem:

Theorem 15. *There exists a three-dimensional grid layout of a butterfly network with* N *inputs and* N *outputs* $\mathscr{B}_N$ *with volume* $\mathrm{O}(N^{1/2}) \times \mathrm{O}(N^{1/2}) \times \mathrm{O}(N^{1/2})$ *and all edges have maximum wire length* $\mathrm{O}(N^{1/4})$, *except* N *edges — all belonging to the same*

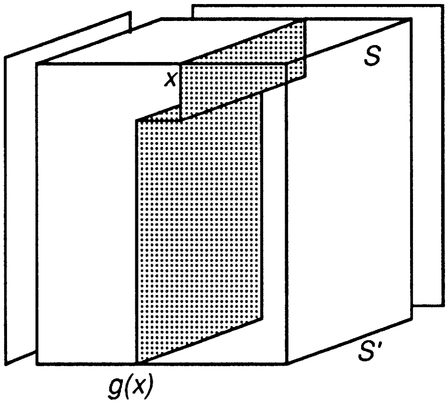


Fig. 6. Three-dimensional double channel routing.

level — having maximum wire length $O(N^{1/2})$.

Notice that the butterfly network is only a representative of a class of isomorphic networks, to which belong, for instance, the Omega network and the Baseline Network.

Of course, the method described to layout butterfly networks is suitable for the whole of the isomorphism class.

3.2. X-tree network

In this subsection we will show how to lay out an N leaf X-tree $\mathscr{T}_N$, where $N = 2^n$, in a three-dimensional grid having $O(N)$ volume, that is optimum. Namely, we construct a drawing of size $O(N^{1/3}) \times O(N^{1/3}) \times O(N^{1/3})$ and maximum wire length $O(N^{1/3})$. To the best of our knowledge, in the literature there are no previous results on this topic.

Our method has some similarities with the method described to lay out a butterfly network, in the sense that we 'cut' an N-leaf X-tree in many smaller X-trees and then we reconnect them together. To this end, from now on, we will assume that n is divisible by 3; if it is not, it is easy to adjust the details, that we omit for brevity.

Observation 2. *The subgraph of* $\mathscr{T}_N$ *induced by nodes at levels* $\log N^{1/3}, \ldots, \log N$ *consists of* $N^{1/3}$ *copies of* $N^{2/3}$*-leaf X-tree networks* $\mathscr{T}_{N^{2/3}}$ *joined by edges connecting the rightmost node of each copy to the leftmost node of the next copy, on the same level* (*see* Fig. 7). *The root of each copy coincides with a leaf of the* $N^{1/3}$*-leaf X-tree — call it X*-tree father *— induced by nodes at level* $0, \ldots, \log N^{1/3}$.

As a consequence of the previous observation, our layout consists of three main operations:

- Three-dimensional layout of each copy of $\mathscr{T}_{N^{2/3}}$;

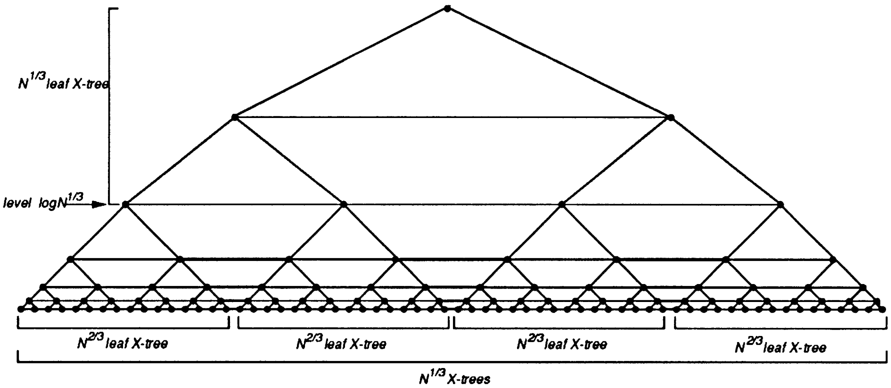


Fig. 7. An X-tree decomposed as union of smaller X-trees.

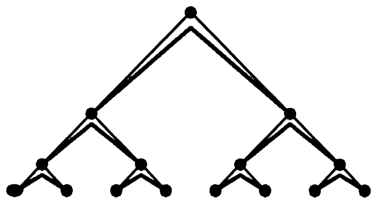


Fig. 8. Non-tree edges visualized as couples of tree edges.

- Re-connection of the $\mathscr{T}_{N^{2/3}}$ through the horizontal edges;
- Layout and re-connection of the X-tree father.

3.2.1. Three-dimensional layout of each copy of $\mathscr{T}_{N^{2/3}}$

From the definition itself of X-tree, we can distinguish in a $\mathscr{T}_K$ a K leaf complete binary tree and a set of $2K-2-\log K$ horizontal non-tree edges. It is easy to lay out the binary tree, as an H-tree on a bidimensional $O(\sqrt{K}) \times O(\sqrt{K})$ grid. From now on we will call π the plane where this H-tree lies.

It is also easy to lay out a part of the set of non-tree edges in view of the following observation:

Observation 3. *Consider the set of $K-1$ non-tree edges of $\mathscr{T}_k$ lying alternately on each level. Each of them can be visualized on a K leaf complete binary tree as a couple of edges connecting two siblings, eliminating their father* (*see Fig.* 8).

It is possible to lay out all such $K-1$ non-tree edges on a new plane π'; to this end, trace a unit length connection orthogonal to π towards π' from the extremes of such edges and draw on π' the required connections. Then, on π' there is a kind of H-tree, whose nodes are substituted by edge crossings.

To manage the set of the remaining non-tree edges, we use an inductive method.

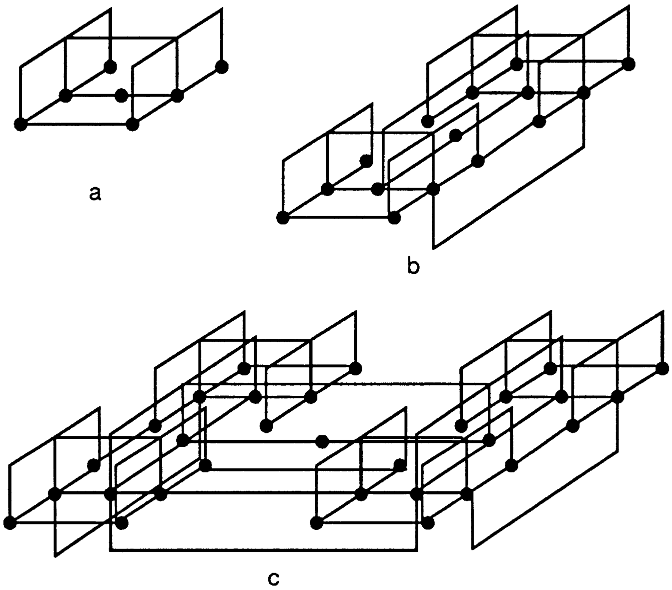


Fig. 9. Three-dimensional drawing of $\mathscr{T}_4$, $\mathscr{T}_8$ and $\mathscr{T}_{16}$.

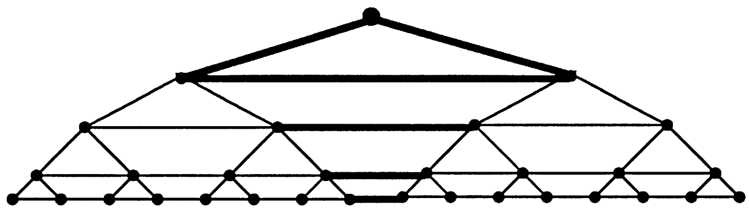


Fig. 10. A $\mathscr{T}_{2K}$ as union of two $\mathscr{T}_K$, a new root and non-tree edges.

The basis of the induction is represented by the three-dimensional drawing of $\mathscr{T}_4, \mathscr{T}_8$ and $\mathscr{T}_{16}$, all depicted in Fig. 9. $\mathscr{T}_4$ and $\mathscr{T}_8$ are initial cases, while $\mathscr{T}_{16}$ is the first X-tree following the inductive rule.

Let K be equal to 2^k and let k be even; if it is not, it is easy to adjust the details. Our claim is that given any $\mathscr{T}_K, K \geqslant 16$, its $2K - 2 - \log K$ non-tree edges can be positioned on the three-dimesional grid in the following way:

a. $K/2$ non-tree edges lie on π;
b. $K - 1$ non-tree edges lie on π';
c. the remaining $K/2 - \log K - 1$ non-tree edges lie on a further plane π'''.

The inductive step consists in considering that each $\mathscr{T}_{2K}$ is constituted by two copies of $\mathscr{T}_K$ connected by a newly introduced root and $\log 2K$ new non-tree horizontal edges (see Fig. 10). Our inductive hypothesis is that $K/2$ edges lie on π, $K-1$ lie on π' and the remaining $K/2 - \log K - 1$ lie on a further plane π''. The $2K$ leaf complete binary

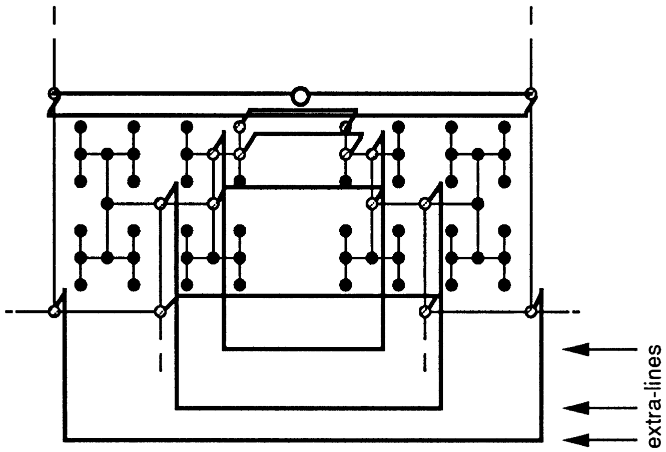


Fig. 11. Edges drawn on π and π''' during the inductive step.

tree inside $\mathscr{T}_{2K}$ can be drawn on π as union of the two K leaf binary trees inside the two copies of $\mathscr{T}_K$ and of the new root.

Let us prove that our claim remains true for $\mathscr{T}_{2K}$ if it is true for $\mathscr{T}_K$:

a. the $K/2+K/2$ non-tree edges of the two $\mathscr{T}_K$ lying on π constitute all non-tree edges of $\mathscr{T}_{2K}$ that must lie on π;
b. the non-tree edge connecting the two children of the root of $\mathscr{T}_{2K}$ takes part in the special H-tree of planes π'; therefore, non-tree edges we put on such planes are $(K-1)+(K-1)$ from the two $\mathscr{T}_K$ plus one, that is $2K-1$;
c. on π'' lie all non-tree edges of the two $\mathscr{T}_K$ lying on it plus all $\log 2K-1$ non-tree edges connecting the two copies of $\mathscr{T}_K$ and not laid yet, that is $2(K/2-\log K-1)+\log 2K-1=K-\log 2K-1$.

It remains to detail how non-tree edges on π'' are drawn. Observe that non-tree edges lying on π'' we add in the inductive phase connect the right-most nodes of a $\mathscr{T}_K$ to the left-most nodes of the other $\mathscr{T}_K$. As far as the H-tree is concerned, we can draw on π'' directly only half of such edges (exactly, $\lceil(\log 2K-1)/2\rceil$); for the remaining non-tree edges we need $\lfloor(\log 2K-1)/2\rfloor$ extra-lines on π'' with respect to the area occupied by the H-tree on π (see Fig. 11). Actually, at each inductive step, it is not necessary to add $\lfloor(\log 2K-1)/2\rfloor$ extra-lines but only one, since we can use the extra-lines introduced in the previous steps.

By following the previous construction, it is possible to express the layout volume of a $\mathscr{T}_K$ by means of a recursive formula, whose solution is

- $3\times(\frac{11}{4}\sqrt{K}-3)\times(\frac{19}{16}\sqrt{K}-3)$ when k is even;
- $2\times(\frac{23}{16}\sqrt{K/2}-3)\times(\frac{35}{16}\sqrt{K/2}-3)$ when k is odd.

Recall that the aim of this step is to lay out each copy of $\mathscr{T}_{N^{2/3}}$. Therefore, all the previous arguments lead to the claim that there exists a three-dimensional grid layout of an $N^{2/3}$ leaf X-tree $\mathscr{T}_{N^{2/3}}$ with volume $O(N^{1/3})\times O(N^{1/3})\times O(1)$ and all edges have maximum wire length $O(N^{1/3})$.

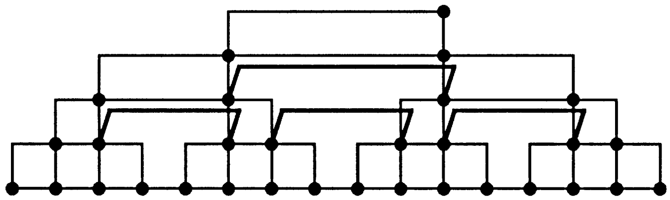


Fig. 12. Layout of the X-tree father.

3.2.2. *Re-connection of the $\mathscr{T}_{N^{2/3}}$ through the horizontal edges*

Observe that each node of a copy of $\mathscr{T}_{N^{2/3}}$, to be connected to the corresponding node of another copy, always has a free direction orthogonal to π and a free direction parallel to π towards the boundary of the layout. Put each of the $N^{1/3}$ copies of $\mathscr{T}_{N^{2/3}}$ one on top of the other, so to build a stack of layouts, having total volume $O(N^{1/3}) \times O(N^{1/3}) \times O(N^{1/3})$. Note that in view of the properties of each layout, characterized by having two free directions coming out from each node to be connected, it is immediate to prove that the addition of two planes, mutually orthogonal and both orthogonal to plane π, is enough to connect all layouts by means of the horizontal edges.

3.2.3. *Layout and re-connection of the X-tree father*

The roots of the $N^{1/3}$ copies of $\mathscr{T}_{N^{2/3}}$ all have the same free direction toward the boundary of the layout of the stack. The intersections of all such free directions with the boundary of the stack lie on a vertical line along a side of the stack. Besides, the roots of the $N^{1/3}$ copies of $\mathscr{T}_{N^{2/3}}$ have a free direction both up and down and therefore they can be connected each other by straight-line edges.

Furthermore, it is easy to lay out the X-tree father on a couple of parallel planes of dimension $N^{1/3} \times (\log N^{1/3} + 1)$ such that the maximum wire length is $(O(N^{1/3})$, as shown in Fig. 12.

Since the leaves of the X-tree father with their horizontal edges and the roots of the $N^{1/3}$ copies connected each other must be merged, it is easy to stretch the layout of the X-tree father and arrange it on one further plane orthogonal to the planes of the all copies.

Therefore, the complete layout of $\mathscr{T}_N, \log N$ divisible by 3, has volume

- $3N^{1/3} \times (\frac{11}{4}N^{1/3} - 2) \times (\frac{19}{16}N^{1/3} - 1)$ when n is even;
- $3N^{1/3} \times (\frac{23}{16\sqrt{2}}N^{1/3} - 2) \times (\frac{35}{16\sqrt{2}}N^{1/3} - 1)$ when n is odd.

In general, the following theorem holds:

Theorem 16. *There exists a three-dimensional grid layout of an N leaf X-tree $\mathscr{T}_N$ with volume* $O(N^{1/3}) \times O(N^{1/3}) \times O(N^{1/3})$ *and all edges have a maximum wire length* $O(N^{1/3})$.

4. Conclusions and open problems

In the first part of this paper we proved a theorem to compute an exact value for the lower bound on the three-dimensional layout volume of any interconnection network. We used this formula to get a lower bound on the layout volume of butterfly and X-tree networks. Besides these networks, we can consider others as data manipulator, Batcher, mesh of trees and multigrid networks, for which we can obtain a lower bound of $\Omega(N^{3/2})$ for the first two networks and $\Omega(N^2)$ for the last ones.

In the second part of this paper we provided a method to optimally lay out in three dimensions butterfly and X-tree networks. Unfortunately, it is not so easy to optimally lay out any network. In fact, the arrangement we could find for data manipulators, Batcher and mesh of trees does not produce an optimal volume. In the layout of a multigrid network we found [4] that some wires have the maximum length $O(N)$. Thus it is an open problem to optimally draw such networks, and to study all the other networks having maximum degree 6.

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