



Antibandwidth of Complete k -Ary Trees

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Abstract

The antibandwidth problem is to label vertices of a n -vertex graph injectively by $1, 2, 3, \dots, n$, such that the minimum difference of labels of adjacent vertices is maximised. The problem is motivated by obnoxious facility location problem, radiocolouring, work and game scheduling and is dual to the well known bandwidth problem. We prove exact results for the antibandwidth of complete k -ary trees, k even, and estimate the parameter for odd k up to the second order term. This extends previous results for complete binary trees.

Keywords: antibandwidth, complete k -ary tree

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1 Introduction

The antibandwidth problem consists of labelling vertices of an n -vertex graph $G = (V, E)$ injectively by $1, 2, 3, \dots, n$, such that the minimum difference of labels of adjacent vertices is maximised. The corresponding maxmin value is denoted by $\text{ab}(G)$. This problem is the dual one of the classical bandwidth problem [3]. It is naturally motivated by obnoxious facility location problems [1], radiocolouring [5] and work and game scheduling tasks [7]. It also belongs to the broad family of graph labelling problems [4]. In the literature it is known under different names: separation number [7], dual bandwidth [9] and antibandwidth [12].

The antibandwidth problem is NP-hard [7]. So far it is known to be polynomially solvable for 3 classes of graphs: the complements of interval, arborescent comparability and threshold graphs [2,6]. Known results include simple relations of the antibandwidth invariant to the minimum, maximum degree, chromatic index and powers of hamiltonian paths in the complement graph [7,9,10]. Exact results and tight bounds are known for paths, cycles, special trees, meshes, hypercubes [9,10,12,13]. The class of n -vertex forests with $\text{ab}(F) = \lfloor n/2 \rfloor$ is characterized in [10], which for complete binary trees gives a value of $(n-1)/2$. The same result for complete binary trees was also independently proved in [13].

In our paper we prove that the antibandwidth of the n -vertex complete k -ary tree, for $k \geq 4$ even, is $(n-k+1)/2$. For odd k , we show tight bounds up to the second order term. In particular, the antibandwidth of the n -vertex complete ternary tree of height h is $n/2 - \Theta(h)$. For $h = 2$ and odd k the antibandwidth equals $(k^2 + 1)/2$.

2 Basic Notions

Let $T(k, n)$ be the n -vertex, complete k -ary tree. We have $n = 1 + k + k^2 + \dots + k^h = (k^{h+1} - 1)/(k - 1)$, where h is the height of the tree. Divide vertices of the tree into $h + 1$ levels according to their distances from the root, which is on level 1. Let $d(v)$ be the degree of a vertex v . Of course $d(v)$ can be either 1 (if v is a leaf), or k (if v is the root) or $k + 1$ (if v is an internal vertex).

For a nonempty graph $G = (V, E)$, let f be a one-to-one labelling $f : V \rightarrow \{1, 2, 3, \dots, |V|\}$. Define the *antibandwidth of G according to f* as

$$\text{ab}(G, f) = \min_{uv \in E} |f(u) - f(v)|.$$

The *antibandwidth of G* is defined as

$$\text{ab}(G) = \max_f \text{ab}(G, f).$$

It is useful to imagine the antibandwidth problem as a linear layout problem. The vertices are mapped into integer points $\{1, \dots, |V|\}$ on a line such that the minimal distance of adjacent vertices is maximised.

We say that a set of vertices U in a graph $G = (V, E)$ is a *vertex r -bisector* if removing U the remaining vertices are partitioned into disjoint sets V_1, V_2 , s.t. $|V_1|, |V_2| \leq r$ and every path between V_1 and V_2 contains a vertex from U .

Similarly, we say that a set of edges F in a graph $G = (V, E)$ is an *edge $\lceil n/2 \rceil$ -bisector* if removing F the vertices are partitioned into disjoint sets V_1, V_2 , s.t. $|V_1|, |V_2| \leq \lceil n/2 \rceil$ and every edge between V_1 and V_2 belongs to F .

3 Even k Case

In this section we will provide the exact value of the antibandwidth of a complete k -ary tree, where k is even.

Theorem 3.1 For even $k \geq 4$,

$$ab(T(k, n)) = \frac{n + 1 - k}{2}.$$

Proof. Lower bound. We prove the lower bound by providing a labelling. The idea is to assign the middle label to the root, to assign the smallest and largest labels to nodes at the first level, and to proceed by assigning labels to the children of a node labelled l by using labels as far as possible from l . So, the root, at level 1, is labelled by $(n + 1)/2$. The vertices at level 2 have labels consecutively from the left $1, 2, 3, \dots, \frac{k}{2}, n - \frac{k}{2} + 1, n - \frac{k}{2} + 2, \dots, n$. The vertices at level 3 have labels consecutively from the left $n - \frac{k}{2} - \frac{k^2}{2} + 1, n - \frac{k}{2} - \frac{k^2}{2} + 2, \dots, n - \frac{k}{2}, \frac{k}{2} + 1, \frac{k}{2} + 2, \dots, \frac{k}{2} + \frac{k^2}{2}$ and so on. As an example, see Figure 1. One can check that the minimum difference of labels is as claimed.

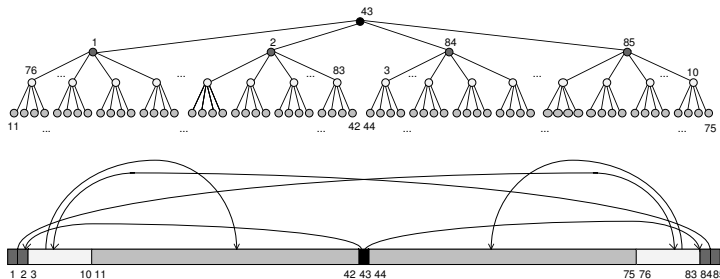


Fig. 1. An example of labeling of a complete 4-ary tree.

Upper Bound. We proceed by contradiction, so let us assume that

$$ab(T(k, n)) \geq \frac{n + 1 - k}{2} + 1.$$

Let $f : V_T \rightarrow \{1, 2, \dots, n\}$ be a bijective labelling of the vertices of $T(k, n)$. Then, two cases can arise:

- (i) There exists a vertex v with neighbours u and w , such that $f(u) < f(v) < f(w)$. Hence $d(v) \geq k$. Then, we can define two integer values l and $r = d(v) - l$, both ≥ 1 such that u_1, u_2, \dots, u_l and w_1, w_2, \dots, w_r are neighbours of v and $f(u_1) < f(u_2) < \dots < f(u_l) < f(v) < f(w_1) < f(w_2) < \dots < f(w_r)$ (for a visualization, see Fig. 1).

It follows that $f(w_1) - f(u_l) \leq n - 1 - (l - 1) - (r - 1) \leq n + 1 - k$ since $l + r \geq k$.

Hence $\min\{f(v) - f(u_l), f(w_1) - f(v)\} \leq \frac{n+1-k}{2}$, a contradiction.

- (ii) For any v with neighbours $u_1, u_2, \dots, u_{d(v)}$ either $f(u_i) < f(v)$, for all $i = 1, 2, \dots, d(v)$ or $f(u_i) > f(v)$, for all $i = 1, 2, \dots, d(v)$. Let I be the interval $[(n + 1 - k)/2, (n + 1 + k)/2]$ and let us focus on the vertices with degree strictly greater than 1.

- (a) Assume there exists v , with $d(v) > 1$, s.t. $f(v) \in I$. W.l.o.g. assume that $f(v) \leq (n + 1)/2$.

If for all neighbours $u_1, u_2, \dots, u_{d(v)}$ of v it holds $f(u_1) < f(u_2) < \dots < f(u_j) < f(v)$,

then

$$f(v) - f(u_j) \leq \frac{n+1}{2} - 1 - (j-1) \leq \frac{n+1-k}{2},$$

a contradiction.

If, on the contrary, for all neighbours $u_1, u_2, \dots, u_{d(v)}$ of v , it holds $f(v) < f(u_1) < f(u_2) < \dots < f(u_j)$ then

$$f(u_1) - f(v) \leq n - \frac{n+1}{2} + \frac{k}{2} - (j-1) \leq \frac{n+1-k}{2},$$

again a contradiction.

- (b) Assume that for all v with $d(v) > 1$, it holds $f(v) \notin I$. Consider the root r . As $f(r) \notin I$, w.l.o.g. assume that $f(r) \leq \frac{n+1-k}{2} - 1$. Then for all vertices w on level 2 we have $f(w) \geq \frac{n+1-k}{2} + 1$. Similarly, for vertices w on level 3 we have $f(w) \leq \frac{n+1-k}{2} - 1$, etc., until we reach the vertices on level h . W.l.o.g. assume that for all vertices w on level h we have $f(w) \geq \frac{n+1-k}{2} + 1$.

As $k^h \geq \frac{n-1+k}{2}$, at least one leaf w satisfies:

$$f(w) \geq \frac{n+1-k}{2} + 1.$$

Clearly, for the parent p of w : $f(w) < f(p)$. Hence

$$f(p) - f(w) \leq n - \frac{n-1+k}{2} = \frac{n+1-k}{2},$$

a contradiction. □

4 Odd k Case

In this section we provide upper and lower bounds for the antibandwidth that differ in a lower order term, in the case k odd. Unfortunately, in this case, the symmetric construction exploited in the even case cannot be applied, so we will use a completely different technique.

Theorem 4.1 *For odd $k \geq 3$ and $h \geq 3$*

$$ab(T(k, n)) \leq \frac{n}{2} - \max\left\{\frac{k}{2}, \frac{h}{8} - o(h)\right\}$$

Proof. Sketch. The upper bound of the form $(n - k)/2$ can be obtained in a similar way as for the k even case. For the second upper bound assume that h is odd. The even h case can be proven similarly. Let S be the smallest set of vertices whose removal divides the vertices of the resulting forest into independent sets X and Y , s.t. $|X|, |Y| \leq n/2$. We claim that

$$ab(T(k, n)) \leq \frac{n - |S|}{2}.$$

To prove this, consider an optimal layout. Removing the last $n - 2ab(T(k, n))$ vertices we get 2 independent sets: the first one is the set on positions $1, 2, 3, \dots, ab(T(k, n))$ and the second one is the set on the positions $ab(T(k, n)) + 1, \dots, 2ab(T(k, n))$. Note that there are possible edges between the two sets only, otherwise we get an edge of length smaller than $ab(T(k, n))$.

As $ab(T(k, n)) \leq n/2$ we have

$$|S| \leq n - 2ab(T(k, n)),$$

which proves the claim.

In what follows we prove that $|S| \geq h/4 - o(h)$. We need some new notations. Let L_i , for $i = 1, 2, 3, \dots, h + 1$ denote the set of vertices of the i -th level of the tree, while L_1 contains the root. Set $x_i = |L_i \cap X|$, $y_i = |L_i \cap Y|$, $s_i = |L_i \cap S|$. Observe that, for $i \geq 2$, as X , Y and S are defined, and in view of the structure of a complete k -ary tree, we have that

$$(1) \quad k(x_{i-1} + y_{i-1} + s_{i-1}) = x_i + y_i + s_i.$$

Furthermore, the properties of X , Y and S imply that the children of vertices in $L_{i-1} \cap X$ must be in $L_i \cap (S \cup Y)$, hence $y_i + s_i \geq kx_{i-1}$. By (1), this is equivalent to $ky_{i-1} + ks_{i-1} \geq x_i$. Repeating this argument for $L_{i-1} \cap Y$ we derive the following:

$$(2) \quad x_i - ks_{i-1} \leq ky_{i-1} \leq x_i + s_i$$

$$(3) \quad y_i - ks_{i-1} \leq kx_{i-1} \leq y_i + s_i$$

Now we show that S is a vertex $(n/2 + 7|S|/2)$ -bisector. It is easy to see that the sets

$$V_1 = \cup_{\text{even } i} (L_i \cap X) \cup \cup_{\text{odd } i} (L_i \cap Y), V_2 = \cup_{\text{odd } i} (L_i \cap X) \cup \cup_{\text{even } i} (L_i \cap Y)$$

are distinct and any path between them contains a vertex from S . Hence S is a vertex r -bisector. Let us estimate r .

$$(4) \quad |V_1| = \sum_{\text{even } i} x_i + \sum_{\text{odd } i} y_i \leq \sum_{\text{even } i} x_i + \frac{1}{k} \sum_{\text{even } i} (x_i + s_i) \leq \frac{k+1}{k} \sum_{\text{even } i} x_i + \frac{1}{k} |S|.$$

To estimate the last sum we need estimations for every x_i , for even i . From the left hand side of inequality (3) we have

$$\begin{aligned} \sum_{i=2}^{h+1} (y_i - ks_{i-1}) &\leq k \sum_{i=2}^{h+1} x_{i-1} \\ |Y| - y_1 - k|S| &\leq k(|X| - x_{h+1}) \\ n - |X| - |S| - y_1 - k|S| &\leq k(|X| - x_{h+1}) \\ kx_{h+1} &\leq (k+1)|X| - n + (k+1)|S| + 1 \leq \frac{(k+1)n}{2} - n + (k+2)|S| \end{aligned}$$

$$(5) \quad x_{h+1} \leq \frac{k-1}{2k}n + \frac{k+2}{k}|S|.$$

Combining right hand sides of inequalities (2) and (3) we have:

$$x_{i-2} \leq \frac{1}{k}(y_{i-1} + s_{i-1}) \leq \frac{1}{k}(\frac{1}{k}(x_i + s_i) + s_{i-1}) = \frac{1}{k^2}(x_i + s_i + ks_{i-1}).$$

Iterating this inequality backwards, starting with $i = h + 1$ we get for even $i \geq 2$

$$x_i \leq \frac{1}{k^{h-i+1}}(x_{h+1} + \sum_{j=i+1}^{h+1} k^{h+1-j} s_j).$$

Using this estimation we compute

$$\begin{aligned} \sum_{\text{even } i \geq 2}^{h+1} x_i &\leq \sum_{\text{even } i \geq 2}^{h+1} \frac{x_{h+1}}{k^{h-i+1}} + \sum_{\text{even } i \geq 2}^{h+1} \sum_{j=i+1}^{h+1} \frac{s_j}{k^{i-j}} \\ &< x_{h+1} \sum_{\text{even } t \geq 0}^{h-1} \frac{1}{k^t} + \sum_{j=3}^{h+1} \left(\frac{1}{k} + \frac{1}{k^3} + \dots + \frac{1}{k^{h-2}} \right) s_j \\ &< x_{h+1} \sum_{\text{even } t \geq 0}^{\infty} \frac{1}{k^t} + \sum_{j=3}^{h+1} \frac{k}{k^2 - 1} s_j \end{aligned}$$

$$< \frac{k^2}{k^2 - 1}x_{h+1} + \frac{k}{k^2 - 1}|S|.$$

Substituting (6) into (4) and using (5) we obtain

$$\begin{aligned} |V_1| &\leq \frac{k}{k-1}x_{h+1} + \frac{2}{k-1}|S| \leq \frac{k}{k-1}\left(\frac{k-1}{2k}n + \frac{k+2}{k}|S|\right) + \frac{2}{k-1}|S| \leq \frac{n}{2} + \frac{k+4}{k-1}|S| \\ &\leq \frac{n}{2} + \frac{7}{2}|S|. \end{aligned}$$

Repeating the same calculations for $|V_2|$ we get the same bound, hence concluding that S is a vertex $(n/2 + 7|S|/2)$ -bisector. Assume $|V_1| \leq |V_2|$. Let $|V_2| = n/2 + p$. By deleting a suitable set of at most $\log_k p + 1$ vertices we can separate p vertices from V_2 and add them to V_1 . To see this, observe that p can be expressed in the form

$$p = \sum_{i=1}^z \alpha_i \frac{k^i - 1}{k - 1},$$

where $0 \leq \alpha_i \leq k$ are integers, and z is the smallest number s.t. $(k^{z+1} - 1)/(k - 1) > p$, i.e., $z \leq \log_k p + 1$. And note that by removing a suitable vertex from V_2 we get k complete subtrees of size $(k^j - 1)/(k - 1)$, where $j \leq z$.

Thus we get a vertex $n/2$ -bisector. Its size is

$$|S| + \log_k p + 1 \leq |S| + \log_k \frac{7}{2}|S| + 1.$$

Further, removing all edges incident to the vertices of the vertex $n/2$ -bisector and distributing the isolated vertices among the current sets V_1 and V_2 in such a way that neither of them contains more than $n/2$ vertices we get an edge $n/2$ -bisector of the size at most

$$(k + 1)(|S| + \log_k \frac{7}{2}|S| + 1).$$

It is known [11] that the size of the smallest edge $\lceil n/2 \rceil$ -bisector of the complete k -ary n -vertex tree of height h is at least

$$\frac{k - 1}{2}(h - \log_k h - 1).$$

Thus we have

$$(k + 1)(|S| + \log_k \frac{7}{2}|S| + 1) \geq \frac{k - 1}{2}(h - \log_k h - 1),$$

Hence

$$|S| \geq \frac{k - 1}{2(k + 1)}(h - \log_k h - 1) - \log_k \frac{7}{2}|S| - 1.$$

As $|S| \leq h$, this yields

$$|S| \geq \frac{k - 1}{2(k + 1)}h - o(h) \geq \frac{h}{4} - o(h).$$

□

In the following paragraphs, for the sake of completeness, we shortly repeat the algorithm by Miller and Pritikin [10]. This algorithm provides reasonably good layout for forests and we use its slight modification in the lower bound construction in the next theorem.

For a bipartite graph B with a specified bipartition M, N with $|M| \leq |N|$, we refer to the *minority* $MIN(B) = |M|$ and *majority* $MAJ(B) = |N|$ of B and refer to M and N as being the minority and majority sides, respectively.

Given any bipartition X, Y of a forest with $|X| \leq |Y|$, there always exists a vertex $y \in Y$ of degree 0 or 1 since the average degree of the majority side vertices is at most $(|X| + |Y| - 1)/|Y|$, which is less than two.

Let a forest F_1 have minority side M_1 and majority side N_1 . For each $i \in [1, MAJ(F_1)]$, recursively define y_i, x_i, M_i, N_i as follows. Let $y_i \in N_i$ be a vertex of degree 0 or 1 in F_i . If y_i has degree 1 in F_i choose x_i as its sole neighbour. If M_i is empty, choose $x_i = y_i$. In any other case, choose x_i to be any element of M_i . Let $F_{i+1} = F_i - x_i - y_i$, $M_{i+1} = M_i - x_i$, $N_{i+1} = N_i - y_i$. The resulting layout is obtained by the following labeling. Assign $f(x_i) = i$ for each $i \in [1, MIN(F_1)]$ and $f(y_i) = MIN(F_1) + i$ for each $i \in [1, MAJ(F_1)]$. This leads to a construction with

$$ab(F) \geq MIN(F)$$

Theorem 4.2 For odd $k \geq 3$ and $h \geq 3$

$$ab(T(k, n)) \geq \frac{n}{2} - O(k^2h)$$

Proof. Sketch. We proceed with the following construction.

- (i) Number the levels of the tree by $1, 2, \dots, h + 1$. For every level $i : i \geq 2$ number the vertices from left to right by integers $1, 2, \dots, k^{i-1}$. Then delete the vertex with label $\lfloor \frac{k^{i-1}}{2} \rfloor$ together with its adjacent edges. Define the set D consisting of deleted vertices. The remaining parts of the tree define the forest F .
- (ii) Divide the vertices of F into two partitions X and Y s.t. $|Y| - 1 \leq |X| \leq |Y|$.
- (iii) For every $v : v \in Y$ such that v was adjacent to some $d \in D$ define the priority to 2. For all neighbours of every such v define the priority value to 1. The rest of vertices of F obtain priority value 3. The higher priority the lower its value.
- (iv) Use the modified Miller/Pritikin algorithm to get the layout of F with $ab(F) \geq \lfloor \frac{n-h}{2} \rfloor$. The modification of used algorithm simply follows the priorities of vertices defined in the previous step. If it is not possible to label a vertex w with priority 1 directly, i.e. the vertex w do not have neighbour from Y of degree 1 or there is no vertex from Y with degree 0, label one of the leaves from Y of degree 1 and its parent from X and remove them from the forest. This operation creates $k - 1$ isolated vertices from Y which can be used for labeling the vertices with priority 1.
- (v) Place the vertices from set D in the middle of the layout, between the sets X and Y .

The algorithm places the vertices from the sets X, Y, D in the order X, D, Y . For the final lower bound the distance from the neighbours of D to D is important. Let P_i be the set of vertices of priority i . Since every deleted vertex except the last one has $(k + 1)$ neighbours, approximately half of them belongs to the set Y , i.e. $|P_2| = (k + 1)h/2$. Every vertex from P_2 has k neighbours from X , i.e. $|P_1| = (k + 1)kh/2$. To label the vertices of P_1 we need $|P_1|$ vertices from Y of degree 0. These can be easily produced from leaves (see step 4 of the algorithm). With a simple analysis we get that the labeling of P_1 needs

$$\frac{hk(k + 1)}{2} \cdot \frac{(k + 1)}{k} = \frac{h(k + 1)^2}{2}$$

leaves. Labeling of P_1 vertices will make all of P_2 vertices from Y isolated and therefore they can be used to label the second half of P_2 vertices from X . In resulting layout there will be $h(k + 1)^2/2$ P_1 vertices, then $h(k + 1)/2$ P_2 vertices. Since P_2 are the neighbours of D , then

$$ab(T(k, n)) \geq n/2 - h(k + 1)^2/2 - h(k + 1)/2 = n/2 - O(hk^2)$$

□

Combining our methods we are able to prove that:

Theorem 4.3 For odd $k \geq 3$ and $h = 2$

$$\text{ab}(T(k, n)) = \frac{k^2 + 1}{2}$$

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