On Three-Dimensional Layout of Interconnection Networks (Extended Abstract) ⋆

Tiziana Calamoneri[1] and Annalisa Massini[2]

[1] Dipartimento di Matematica and Dipartimento di Scienze dell'Informazione, Università di Roma "La Sapienza", Italy - calamo@dsi.uniroma1.it.
[2] Dipartimento di Scienze dell'Informazione, Università di Roma "La Sapienza", Italy - massini@dsi.uniroma1.it.

Abstract. In this paper we deal with the layout of interconnection networks on three-dimensional grids. In particular, in the first part we prove a general formula for calculating an exact value for the lower bound on the volume. Then we introduce the new notion of k-3D double channel routing and we use it to exhibit an optimal three-dimensional layout for butterfly networks. Finally, we show a method to lay out multigrid and X-tree networks in optimal volume.

1 Introduction and Preliminaries

Recent hardware advances have allowed three-dimensional circuits to have a cost low enough to make them commonly available. For this reason three-dimensional layouts of graphs on rectilinear grids are becoming of wide interest both in the study of the VLSI layout problem for integrated circuits and in the study of algorithms for drawing graphs. Indeed, the tie between VLSI layout studies and theoretical graph drawing is very strong since to lay out a network on a grid is equivalent to orthogonally draw the underlying graph.

To the best of our knowledge, not many papers have been written about three-dimensional grid drawing of graphs [2–6, 10] and all of them show results that are valid for very general graphs and therefore they do not work efficiently for structured and regular graphs such as the most commonly used interconnection networks. On the other hand, the importance of representing interconnection networks in three dimensions has already been stated in the 80's by Rosenberg [12]: the most relevant aims are to shorten wires and to save in material.

By virtue of the equivalence between layout of networks and drawing of graphs in the following we will prefer the network terminology instead of the graph theory one; therefore we will use the word 'node' instead of 'vertex' and 'layout' instead of 'drawing', while we will interchangeably use the terms 'graph' and 'network', 'edge' and 'wire'.

In this paper we focus our attention on *three-dimensional grid layout* of an interconnection network $\mathcal{G}$, that is a mapping of $\mathcal{G}$ in the three-dimensional grid such that nodes are mapped in grid-nodes and edges are mapped in independent grid-paths satisfying the following conditions:

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- distinct grid-paths are edge-disjoint (then at most three paths can cross at a grid-node);
- grid-paths that share an intermediate grid-node must cross at that node (that is 'knock-knee' paths [9] are not allowed);
- a grid-path may touch no mapped node, except at its endpoints.

If the layout of a graph $\mathcal{G}$ can be enclosed in a $h \times w \times l$ three-dimensional grid, we say *layout volume* of $\mathcal{G}$ the product $h \times w \times l$.

In this work we give some results about lower and upper bounds on the layout volume of some interconnection networks. Namely, in the first part we prove a general formula for calculating an exact value for the lower bound on the three-dimensional layout volume. Then we introduce the new notion of k-3D double channel routing and we use it to exhibit an optimal three-dimensional layout for butterfly networks. Finally, we show a method to lay out multi-grid and X-tree networks in optimal volume.

2 Lower bound

In this section we prove a general formula giving an exact value for the lower bound on the layout volume of interconnection networks.

We obtain our result by generalizing to three dimensions the classical lower bound strategy for two dimensions invented in [13], and modified in [1]. In [12] the order of magnitude of the result obtained in Lemma 2 is given for reticulated graphs and extended to more general graphs. Before proving the general formula for the lower bound, we give some definitions and prove some preliminary results.

Definition 1. *An* embedding *of graph $\mathcal{G}$ into graph $\mathcal{H}$ (which has at least as many nodes as $\mathcal{G}$) comprises a* one-to-one association α *of the nodes of $\mathcal{G}$ with nodes of $\mathcal{H}$, plus a* routing ρ *which associates each edge $\{u, v\}$ of $\mathcal{G}$ with a path in $\mathcal{H}$ that connects nodes $\alpha(u)$ and $\alpha(v)$. The* congestion *of embedding $\langle \alpha, \rho \rangle$ is the maximum, over all edges e in $\mathcal{H}$, of the number of edges in $\mathcal{G}$ whose ρ-routing paths contain edge e.*

Definition 2. *Let $\mathcal{G}$ be a graph having a designated set of $2c > 0$ nodes, called* special nodes. *The* minimum special bisection width *of a graph $\mathcal{G}$, MSBW($\mathcal{G}$), is the smallest number of edges whose removal partitions $\mathcal{G}$ into two disjoint subgraphs, each containing half of $\mathcal{G}$'s special nodes.*

Lemma 1. *[8] Let ϵ be an embedding of graph $\mathcal{G}$ into graph $\mathcal{H}$ that has congestion C, then $MSBW(\mathcal{H}) \geq \frac{1}{C} MSBW(\mathcal{G})$.*

Now we prove a general formula to get a lower bound on the layout volume of a network, given its MSBW.

Lemma 2. *For any graph $\mathcal{H}$, the volume of the smallest three-dimensional layout of $\mathcal{H}$ is at least $\left(\frac{MSBW(\mathcal{H})-1}{2}\right)^{3/2}$.*

Sketch of proof. We consider an arbitrary layout of $\mathcal{H}$ in the grid of dimension $h \times w \times l$. A surface S with a single jog J (see Fig. 1) can be positioned on the grid in such a way that it cuts the layout of $\mathcal{H}$ into two subgraphs, each containing half of $\mathcal{H}$'s special nodes.

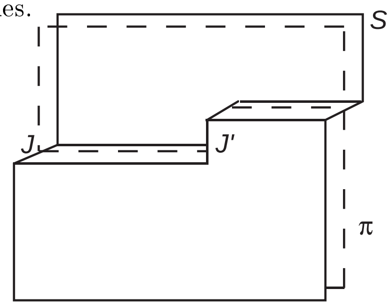

Fig. 1. Surface S with the jog J.

Removing the grid-edges crossed by S yields a bisection of $\mathcal{H}$. By definition, at least $MSBW(\mathcal{H})$ edges of $\mathcal{H}$ must cross surface S. By construction, at most $hl + l + 1 \leq 2hl + 1$ edges of the grid cross surface S. It follows that $2hl + 1 \geq MSBW(\mathcal{H})$. On the other hand, without loss of generality, we can choose $w \geq h \geq l$. Hence $h \times w \times l \geq \left(\frac{MSBW(\mathcal{H})-1}{2}\right)^{3/2}$.

As a consequence of Lemmas 2 and 1, if $MSBW(\mathcal{H})$ is not known, a lower bound on the layout volume of a network $\mathcal{H}$ can be computed through an embedding ϵ into $\mathcal{H}$ of a graph $\mathcal{G}$ if $MSBW(\mathcal{G})$ and the congestion C of ϵ are known. In this way, we have that a lower bound on the layout volume of $\mathcal{H}$ is no less than $\left(\frac{MSBW(\mathcal{H})-1}{2}\right)^{3/2} \geq \left(\frac{\frac{1}{C}MSBW(\mathcal{G})-1}{2}\right)^{3/2}$.

Since another lower bound on the layout volume of a graph is trivially given by the number of nodes of the graph, the following theorem derives:

Theorem 1. *Given a graph $\mathcal{H}$ with n nodes, a lower bound on its layout volume is given by* $\max\left\{n, \left(\frac{MSBW(\mathcal{H})-1}{2}\right)^{3/2}\right\}$. *Alternatively, when an embedding of congestion C for an auxiliary graph $\mathcal{G}$ into $\mathcal{H}$ and $MSBW(\mathcal{G})$ are known, a lower bound on the layout volume of $\mathcal{G}$ is* $\max\left\{n, \left(\frac{\frac{1}{C}MSBW(\mathcal{G})-1}{2}\right)^{3/2}\right\}$.

3 Upper bound of some Interconnection Networks

In this section we first give the definitions of all networks we are going to manage, then we exhibit a method to lay out each of them in a three-dimensional grid.

Definition 3. *The* butterfly network *having N inputs $\mathcal{B}_N$, where $N = 2^n$, has nodes corresponding to pairs $\langle w, l\rangle$ where l is the level $(1 \leq l \leq \log N + 1)$ and w is a $\log N$-bit binary number that denotes the column of the node. Two nodes $\langle w, l\rangle$ and $\langle w', l'\rangle$ are linked by an edge if and only if $l' = l + 1$ and either:*

1. w and w' are identical (straight-edge), or
2. w and w' differ in precisely the l-th bit (cross-edge).

Lemma 3. *[7] The subgraph of $\mathcal{B}_N$ induced by the nodes of levels $1, \ldots, h$ is the disjoint sum of $2^{\log N - h + 1}$ copies of $\mathcal{B}_{2^{h-1}}$ and the subgraph of $\mathcal{B}_N$ induced by the nodes of levels $h, \ldots, \log N + 1$ is the disjoint sum of 2^{h-1} copies of $\mathcal{B}_{2^{\log N - h + 1}}$.*

Definition 4. *The $N \times N$* multigrid network *$\mathcal{M}_N$, where $N = 2^n$, consists of $\log N + 1$ bidimensional arrays, each one of size $N/2^k \times N/2^k$ for $0 \leq k \leq \log N$. The arrays are interconnected so that node (i, j) on the $2^k \times 2^k$ array is connected to node $(2i, 2j)$ on the $2^{k+1} \times 2^{k+1}$ array for $1 \leq i, j \leq 2^k$, and $0 \leq k < \log N$.*

Definition 5. *The N-leaf* X-tree *$\mathcal{T}_N$, where $N = 2^n$, is a complete N-leaf binary tree with edges added to connect consecutive nodes on the same level of the tree.*

We can utilize Theorem 1 to compute, in particular, a lower bound on the layout volume of the interconnection networks just defined:

- a lower bound on the layout volume of a butterfly network $\mathcal{B}_N$ is $(\frac{N-1}{2})^{3/2}$ and can be obtained by considering the embedding described in [1].
- the number of nodes of a multigrid $\mathcal{M}_N$ constitutes a lower bound on its layout volume, that is $\frac{4N^2-1}{3}$. Indeed, the formula involving $MSBW(\mathcal{M}_N) = \Theta(N)$ produces a worse value.
- similar considerations hold for an N-leaf X-tree $\mathcal{T}_N$, whose $MSBW$ is $\Theta(\log N)$, and therefore a lower bound on its layout volume is $2N - 1$.

For what concerns the upper bound on the layout volume we divide the next part into three subsections, one for each network.

3.1 Butterfly Network

It is easy to obtain an optimal three-dimensional layout of a butterfly network by using the forerunner intuition of Wise [14] used to better visualize a butterfly network in the space. This idea is based on opportunely putting and connecting in the space $O(\sqrt{N})$ copies of any bidimensional optimal layout of a butterfly with $O(\sqrt{N})$ inputs (possible in view of Lemma 3). A drawback of such a nice layout is that the maximum wire length is $O(\sqrt{N})$, and most of the wires reach this upper bound.

In the following we will describe a method to lay out a $\mathcal{B}_N$ in the three-dimensional grid so that all its wires have maximum length $O(N^{1/4})$ but one (additive) edge-level characterized by having maximum wire length $O(\sqrt{N})$.

From now on, we will assume that $\log N$ is even; when $\log N$ is odd it is easy to adjust the details, that we omit for the sake of brevity.

In view of Lemma 3 we can 'cut' $\mathcal{B}_N$ along its median node-level and get $\sqrt{N}$ copies of $\mathcal{B}_{\sqrt{N}}$ (O-group) whose output nodes must be re-connected to the input nodes of other $\sqrt{N}$ copies of $\mathcal{B}_{\sqrt{N}}$ (I-group) through an additive edge-level.

Hence, our layout consists of two main operations:

- three-dimensional layout of each copy of $\mathcal{B}_{\sqrt{N}}$;
- re-connection of the two groups of $\sqrt{N}$ copies of $\mathcal{B}_{\sqrt{N}}$ through an additive edge-level.

Three-dimensional layout of each copy of $\mathcal{B}_{\sqrt{N}}$
In order to explain how to manage this operation, we need to mark the following observation:

Observation 1 *An N-input butterfly network $\mathcal{B}_N$ can be covered by N edge-disjoint complete binary trees as follows:*
- *for any $i = 2, \ldots, \log N$, there are $2^{\log N - i}$ trees T_i having i levels, sharing their leaves with some tree $T_j, j > i$, and their internal nodes with some $T_k, k < i$;*
- *there are two trees $T_{\log N+1}$ having $(\log N + 1)$ levels, sharing their leaves each other, and their internal nodes with some $T_k, k < \log N + 1$.*

An example of this covering for $\mathcal{B}_{16}$ is depicted in Fig. 2.

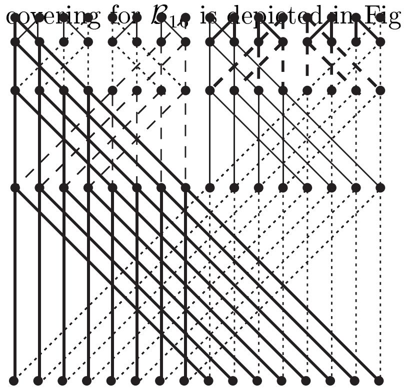

Fig. 2. Tree-covering of $\mathcal{B}_{16}$ (different trees are represented by different line types).

Consider an H-tree representation of $T_{\log\sqrt{N}+1}$, call it $H_{\log\sqrt{N}+1}$. Call H_i a plane representation of T_i obtained from $T_{\log\sqrt{N}+1}$ by eliminating superfluous $\log\sqrt{N} + 1 - i$ levels. Then T_i is represented according to an H-tree scheme wasting some area. Observe that if the leaves of a tree T_j coincide with some internal nodes of a tree $T_i, i > j$, it is possible to lay out T_i and T_j in the three-dimensional grid by considering H_i and H_j on two parallel planes, such that the orthogonal projection of H_j on the plane containing H_i coincides, level by level, with H_i itself. To correctly connect H_i and H_j we have to connect duplicate nodes by a segment orthogonal to both planes and to eliminate the leaves of H_j, substituting them with bends (see Fig. 3).

In view of Observation 1, it remains to detail in which order the planes containing the $\sqrt{N}$ binary trees must be arranged. The following recursive pseudocode allows one to assign a z-coordinate to each plane containing T_j ($z \leftarrow T_j$ for short). The first call of the procedure is PUT($T_{\log\sqrt{N}+1}$, 0).

PROCEDURE PUT(T_j, VAR z);
BEGIN

```
    z ← T_j;
    z + 1 ← (T_2 sharing its leaves with level 2 of T_j);
    i := 3;
    WHILE (i < j) DO
    BEGIN
      PUT(T_i sharing its leaves with level i of T_j, z + 2);
      i:=i+1;
    END;
END.
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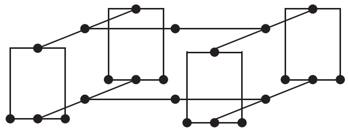

Fig. 3. Layout of two trees sharing some nodes.

After the procedure is terminated, half of $B_{\sqrt{N}}$ has been lain out. The remaining part can be symmetrically laid out in such a way that the planes containing trees $T_{\log\sqrt{N}+1}$ are consecutive.

As far as the procedure is concerned, vertical lines are guaranteed:

- not to cross tree-nodes of intermediate planes; indeed, the procedure puts the trees connected to a certain tree T_j such that as smaller they are as closer to T_j they are positioned;
- not to coincide with other vertical lines; indeed, no more than two trees can share the same nodes.

In view of the construction of the three-dimensional layout of $\mathcal{B}_{\sqrt{N}}$, of Observation 1 and of the area of an H-tree, each butterfly $\mathcal{B}_{\sqrt{N}}$ belonging both to the O-group and to the I-group take a $(2N^{1/4}-1)\times(2N^{1/4}-1)\times(N^{1/2})$ volume.

Re-connection between the two groups of $\sqrt{N}$ copies of $\mathcal{B}_{\sqrt{N}}$

Let us consider the two groups of $\sqrt{N}$ copies of $\mathcal{B}_{\sqrt{N}}$. Each group is positioned in the space to form a square with $N^{1/4}$ copies on each side, such that the correspondent trees of each copy lie on the same plane. The two groups are then positioned one in front of the other. Now we have to connect the duplicated nodes through an additive edge-level.

Before detailing this operation, we need to remind some known results. A *k-channel routing* involves a bidimensional grid and two sets S and S' each consisting of k nodes to be connected by a 1-1 function. S and S' are arranged on two opposite sides of the grid.

Lemma 4. *[11] The grid involved in any k-channel routing is not greater than $(k+1)\times(\frac{3}{2}k+2)$ and S and S' lie on the shorter sides.*

Coming back to the butterfly problem, observe that all the output nodes of the O-group and all the input nodes of the I-group can be provided of an outgoing link towards the opposite group and their extremes can be leaded to two parallel planes, having empty intersection with the layouts of each copy. If we number in the same way –from left to right, row by row– both the output nodes of any butterfly of the O-group and the input nodes of any butterfly of the I-group and the butterflies themselves of O- and I-groups, then each edge must connect the i-th output node of the j-th butterfly in the O-group to the j-th input node of the i-th butterfly in the I-group. Furthermore, it is easy to see that each row of output nodes in the O-group is routed to a row of input nodes in the I-group.

In order to solve this problem we define a new three-dimensional constrained routing, called k-3D double channel routing, to which we reduce the previous problem.

Definition 6. *A* k-3D double channel routing *involves a three-dimensional grid (the channel) and two sets S and S', both of k nodes, to be connected by a 1-1 function f. S and S' are arranged on two opposite sides of the three-dimensional grid, on the nodes of a $\sqrt{k} \times \sqrt{k}$ grid. Function f associates to a node (x, y) of S a node (x', y') of S' such that $x' = g(x)$ and $y' = h(y)$, where functions g and h are two-dimensional $\sqrt{k}$-channel routings.*

Theorem 2. *A three-dimensional grid of size $(\sqrt{k}+1) \times (\sqrt{k}+1) \times (\frac{3}{2}\sqrt{k}+2)$ is enough to realize a k-3D double channel routing.*

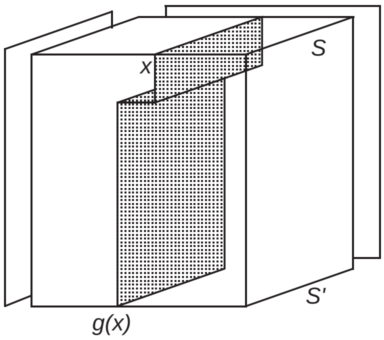

Fig. 4. Three-dimensional double channel routing.

Proof. Project the three-dimensional grid of the k-3D double channel routing on plane xz. It is easy to see that function g mapping rows of S in rows of S' can be considered as a two-dimensional channel routing on plane xz. Therefore, a $(\sqrt{k}+1) \times (\frac{3}{2}\sqrt{k}+2)$ two-dimensional grid is enough to realize such a channel routing (Lemma 4). When coming back to three dimensions, lines laid out to represent function g become (bent) planes. Each of such planes has on opposite horizontal sides a row x of S and its corresponding row $g(x)$ of S' and it is at least $\frac{3}{2}\sqrt{k} + 2$ long (see Fig. 4). Therefore, on each plane we can realize a two-dimensional channel routing given by function h, simply by adding an extra-plane, parallel to plane xz.

We use this theorem to lay out the additive edge-level between the O-group and the I-group in at most $\frac{3}{2}\sqrt{N} + 2$ height.

Recombining all the arguments about the volume needed by the two operations of laying out each copy of $\mathcal{B}_{\sqrt{N}}$ and re-connecting the two groups of $\sqrt{N}$ copies of $\mathcal{B}_{\sqrt{N}}$, we can state the following theorem:

Theorem 3. *There exists a three-dimensional grid layout of a butterfly network with N inputs and N outputs $\mathcal{B}_N$ with volume $(2N^{1/2} - N^{1/4} + 1) \times (2N^{1/2} - N^{1/4} + 1) \times (\frac{7}{2}N^{1/2} + 2)$ and all edges have maximum wire length $O(N^{1/4})$, except N edges having maximum wire length $O(N^{1/2})$.*

3.2 Multi-Grid Network

In this subsection we will show how to lay out an $N \times N$ multigrid $\mathcal{M}_N$ in a three-dimensional grid of size $O(N) \times O(N) \times O(1)$ and maximum edge length $O(N)$. It remains an open problem to find an equal sided three-dimensional layout such that the maximum wire length is shortened.

First, we describe how to lay out all the bidimensional arrays (shortly arrays, where no confusion arises), then we show how to connect adjacent arrays.

All nodes and edges of all the arrays can be positioned on a unique plane π in the following way (see Fig. 5):

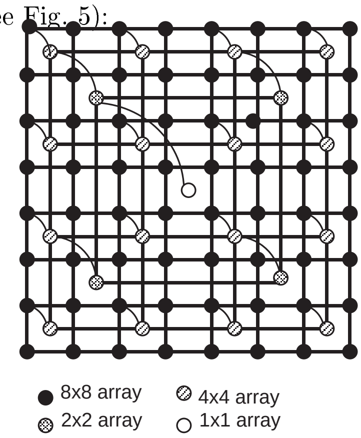

Fig. 5. How to lay out all the bidimensional arrays in a $\mathcal{M}_8$.

- put all nodes of the $N \times N$ array at even coordinates, and connect them in the oblivious way;
- let v_k be the generic node on the $N/2^k \times N/2^k$ array. Put it at coordinates $(x+2^{k-1}, y+2^{k-1})$, where (x, y) are the coordinates of node v_{k-1} to which v_k is connected. Finally, lay out the edges of the current array in the oblivious way.

Edges connecting different arrays can be lain out as follows:

- from any node $v_k, 0 \leq k \leq \log N - 1$ that is endpoint of an edge towards a v_{k+1}, draw a unit length segment orthogonal to π going to an upper plane π' (u-lines);

- from any node $v_k, 1 \leq k \leq \log N$ that is endpoint of an edge towards a v_{k-1}, draw a broken line composed by: a unit length segment orthogonal to π going to a lower plane π'', a unit length segment along y coordinate on π'', and a segment orthogonal to π, going from π'' to π' (l-lines);
- on π', connect the endpoints of the u- and l-lines corresponding to the same edge by means of an L-like line.

Observe that, in view of the position of the nodes on π, both all these edges never cross any node and no collisions arise on π''.

It is easy to see that the area occupied on π by all the arrays is $(2N-1) \times (2N-1)$ and that the addition of π' and π'' is enough to lay out all the remaining edges. Furthermore, the longest wires on π are N long (they belong to the 2×2 array); the longest edge connecting adjacent arrays connects the 2×2 and the 1×1 arrays and is $N + 4$ long. All these considerations lead to the following result:

Theorem 4. *There exists a three-dimensional grid layout of an $N \times N$ multigrid $\mathcal{M}_N$ with volume $(2N-1) \times (2N-1) \times 3$ and all edges have maximum wire length $O(N)$.*

3.3 X-tree Network

In this subsection we will show how to lay out an N leaf X-tree $\mathcal{T}_N$ in a three-dimensional grid having $O(\sqrt{N}) \times O(\sqrt{N}) \times O(1)$ volume, that is optimum. The authors are going to prove that it is possible to lay out an N leaf X-tree in an equal sided three-dimensional grid, such that the maximum wire length is $N^{1/3}$ instead of $\sqrt{N}$.

From the definition itself of X-tree, we can distinguish in a $\mathcal{T}_N$ an N leaf complete binary tree and a set of $2N-2-\log N$ horizontal non-tree edges. It is easy to lay out the binary tree, as an H-tree on a bidimensional $O(\sqrt{N}) \times O(\sqrt{N})$ grid. From now on we will call π the plane where the H-tree lies.

It is also easy to lay out a part of the set of non-tree edges in view of the following observation:

Observation 2 *Consider the set of $N-1$ non-tree edges lying alternately on each level. Each of them can be visualized on an N leaf complete binary tree as a couple of edges connecting two siblings, eliminating their father. See Fig. 6.*

It is possible to lay out all such $N-1$ non-tree edges on a new plane π'; to this end, lead a unit length connection orthogonal to π towards π' from the extremes of such edges and lay out on π' the required connections. Then, on π' there is a kind of H-tree, whose nodes are substituted by knock-knees. We can eliminate them by using two parallel planes, π' and π'', instead of one.

To manage the set of the remaining non-tree edges, we use an inductive method.

Our claim is that given any $\mathcal{T}_k$, $k \geq 4$ and k power of two, its $2k-2-\log k$ non-tree edges can be positioned on the three-dimensional grid in the following way:

a. $k/2$ non-tree edges lie on π;
b. $k-1$ non-tree edges lie on π' and π'' (on a unique plane in Fig. 7);
c. the remaining $k/2 - \log k - 1$ non-tree edges lie on a further plane π'''.

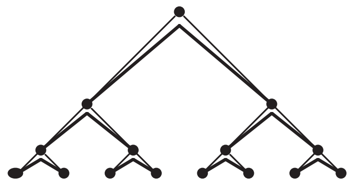

Fig. 6. Non-tree edges visualized as couples of tree edges.

The basis of the induction is represented by the three-dimensional layouts of $\mathcal{T}_4, \mathcal{T}_8$ and $\mathcal{T}_{16}$, all depicted in Fig. 7. $\mathcal{T}_4$ and $\mathcal{T}_8$ are initial cases, while $\mathcal{T}_{16}$ is the first X-tree following our claim.

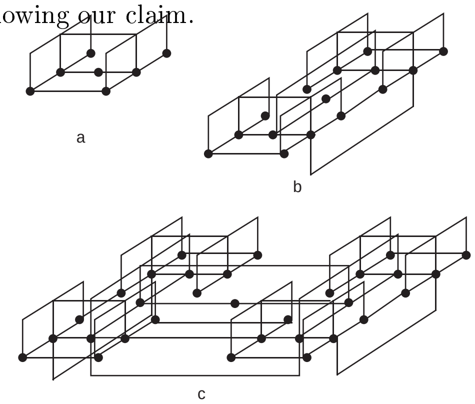

Fig. 7. Three-dimensional layout of $\mathcal{T}_4, \mathcal{T}_8$ and $\mathcal{T}_{16}$.

The inductive step consists in considering that each $\mathcal{T}_N$ is constituted by two copies of $\mathcal{T}_{N/2}$ connected by a newly introduced root and $\log N$ new non-tree horizontal edges (see Fig. 8). Our inductive hypothesis is that $N/4$ edges lie on π, $N/2-1$ lie on π' and π'' and the remaining $N/4 - \log N/2 - 1$ lie on a further plane π'''. The N leaf complete binary tree inside $\mathcal{T}_N$ can be laid out on π as union of the two $N/2$ leaf binary trees inside the two copies of $\mathcal{T}_{N/2}$ and of the new root.

Let us prove that our claim remains true for $\mathcal{T}_N$ if it is true for $\mathcal{T}_{N/2}$:

a. the $N/4+N/4$ non-tree edges of $\mathcal{T}_{N/2}$ lying on π constitute all non-tree edges of $\mathcal{T}_N$ that must lie on π;
b. the non-tree edge connecting the two children of the root of $\mathcal{T}_N$ takes part in the special H-tree of planes π' and π''; therefore, non-tree edges we put on such planes are $(N/2-1)+(N/2-1)$ from the two $\mathcal{T}_{N/2}$ plus one, that is $N-1$;
c. on π''' lie all non-tree edges of the two $\mathcal{T}_{N/2}$ lying on it plus all $\log N - 1$ non-tree edges connecting the two copies of $\mathcal{T}_{N/2}$ and not laid yet, that is $2(N/4 - \log N/2 - 1) + \log N - 1 = N/2 - \log N - 1$.

It remains to detail how non-tree edges on π''' are settled. Observe that non-tree edges lying on π''' we add in the inductive phase connect the right-most nodes of a $\mathcal{T}_{N/2}$ to the left-most nodes of the other $\mathcal{T}_{N/2}$. As far as the H-tree is concerned, we can lay out on π''' directly only $\lceil \frac{\log N - 1}{2} \rceil$ of such edges; for the remaining non-tree edges we need $\lfloor \frac{\log N - 1}{2} \rfloor$ extra-lines on π''' with respect to the area occupied by the H-tree on π (see Fig. 9).

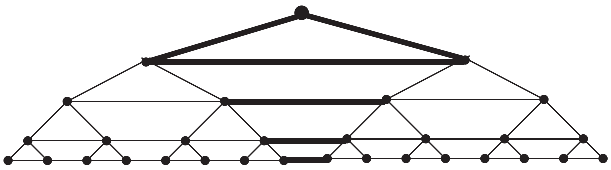

Fig. 8. A $\mathcal{T}_N$ as union of two $\mathcal{T}_{N/2}$ and non-tree edges.

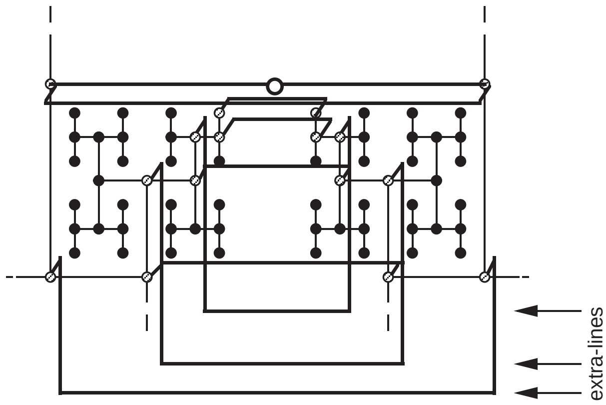

Fig. 9. Edges laid out on π and π''' during the inductive step.

Actually, at each inductive step, it is not necessary to add $\lfloor\frac{\log N-1}{2}\rfloor$ extra-lines but only one, since we can use the extra-lines introduced in the previous steps. Possible knock-knees on π''' can again be avoided by means of a further parallel plane.

By following the previous construction, it is possible to express the layout volume of a $\mathcal{T}_N$ by means of a recursive formula, whose solution is:

- $5 \times (\frac{11}{4}\sqrt{N} - 3) \times (\frac{19}{16}\sqrt{N} - 3)$ when $\log N$ is even;
- $5 \times (\frac{23}{16}\sqrt{N/2} - 3) \times (\frac{35}{16}\sqrt{N/2} - 3)$ when $\log N$ is odd.

All the previous arguments lead to the following result:

Theorem 5. *There exists a three-dimensional grid layout of an N leaf X-tree $\mathcal{T}_N$ with volume $O(\sqrt{N}) \times O(\sqrt{N}) \times O(1)$ and all edges have maximum wire length $O(N^{1/2})$.*

Unfortunately, we did not succeed in applying our inductive method to the three-dimensional version of the H-tree introduced in [12], without increasing the volume of a non-constant factor. It would have implied an optimal layout in an equal sided volume with optimal wire length, that is $O(N^{1/3})$.

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