Algebraic models to improve ranking and query expansion

Latent Semantic Indexing, Word Embeddings

Is there anything more advanced than co-occurrences to learn word correlations?

- Traditional IR uses Term matching, → # of times the doc says “Albuquerque” – not fully appropriate
- We can use a **different approach**: compare all-pairs of query-document **terms**, → # of terms in the doc that relate to Albuquerque
- To detect these similarities:
  - **Latent Semantic Indexing**
  - Word embeddings (a.k.o. deep method – emerging technology)

Albuquerque is the most populous city in the U.S. state of New Mexico. The high-altitude city serves as the county seat of Bernalillo County, and it is situated in the central part of the state, straddling the Rio Grande. The city population is 557,169 as of the July 1, 2014, population estimate from the United States Census Bureau, and ranks as the 32nd-largest city in the U.S. The Metropolitan Statistical Area (or MSA) has a population of 902,797 according to the United States Census Bureau's most recently available estimate for July 1, 2013.

Passage *about* Albuquerque

Allen suggested that they could program a BASIC interpreter for the device; after a call from Gates claiming to have a working interpreter, MITS requested a demonstration. Since they didn't actually have one, Allen worked on a simulator for the Altair while Gates developed the interpreter. Although they developed the interpreter on a simulator and not the actual device, the interpreter worked flawlessly when they demonstrated the interpreter to MITS in Albuquerque, New Mexico in March 1975; MITS agreed to distribute it, marketing it as Altair BASIC.

Passage *not about* Albuquerque

The problem

- With the standard term-document matrix encoding, each term is a vector and dimensions are documents
- Different terms have no inherent similarity, e.g.:

  Search: [0 2 0 0 0 0 0 0 0 1 0 0 0 0]

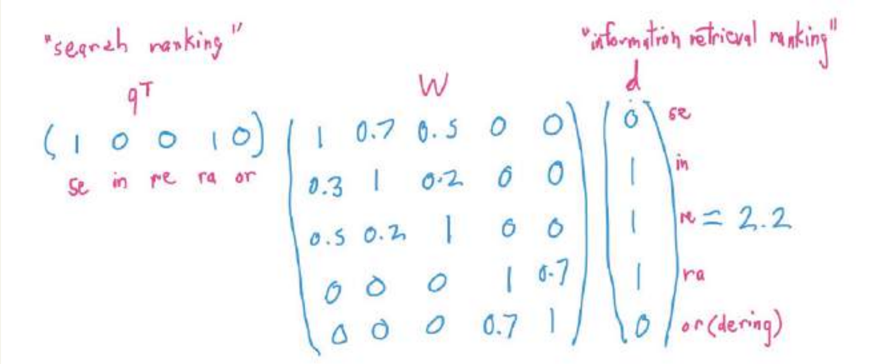
  Information retrieval:[0 0 0 0 0 0 0 3 0 0 0 0 1 0 0]
- If query on *search* and document has *information retrieval* , then our query and document vectors are orthogonal. **Dot product is zero**. But these two words are very related!

Can we directly learn term relations?

- Basic IR is scoring on $\mathbf{q}^T\cdot\mathbf{d}/\mathbf{K}$ (dot product of query and document vectors)

$$\frac{\vec{q}\bullet\vec{d}}{|\vec{q}||\vec{d}|}$$

- No treatment of synonyms; no machine learning
- Can we learn a matrix **W** to rank via $\mathbf{q}^T\mathbf{W}\mathbf{d}$, rather than $\mathbf{q}^T\cdot\mathbf{d}$?



- Where W is a matrix that captures **similarity between words** (e.g., "search" and "information retrieval")?

Latent Semantic Indexing

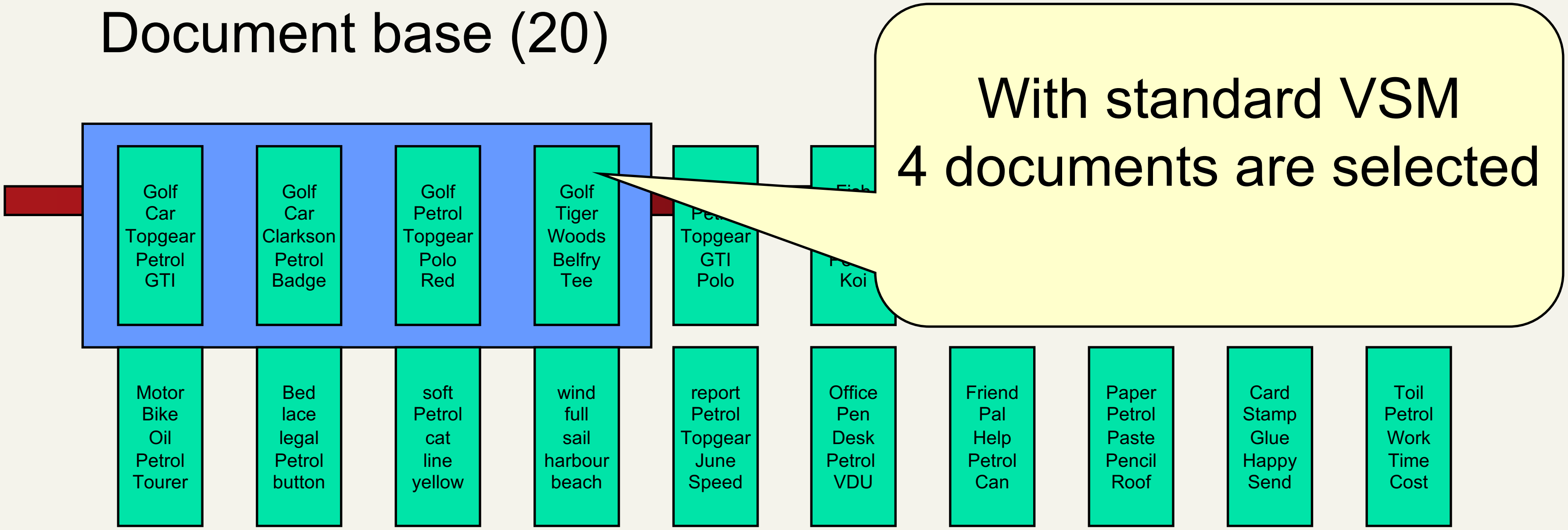
Latent Semantic Indexing

- Term-document matrices are very large, though most cells are “zeros”
- But the number of **topics** that people talk about is smaller (in some sense)
  - Clothes, movies, politics, …
  - Each topic can be represented as a cluster of (semantically) related terms, e.g.: clothes=golf, jacket, shoe..
- Can we represent the term-document space by a lower dimensional “latent” space (**latent space=set of topics**)?

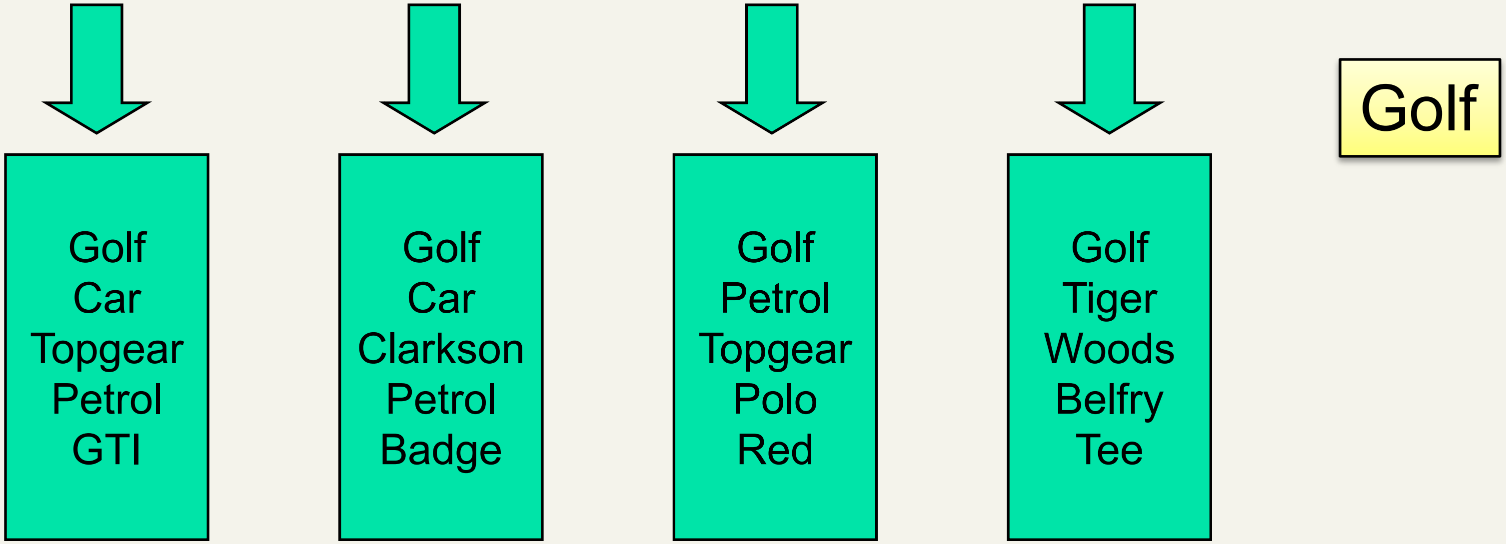
Searching with latent topics

- Given a collection of documents, LSI learns clusters of frequently co-occurring terms (ex: `information retrieval`, `ranking` and `web`)
- If you query with `ranking`, **`information retrieval`** LSI "automatically" extends the search to documents including also (and even ONLY) `web`

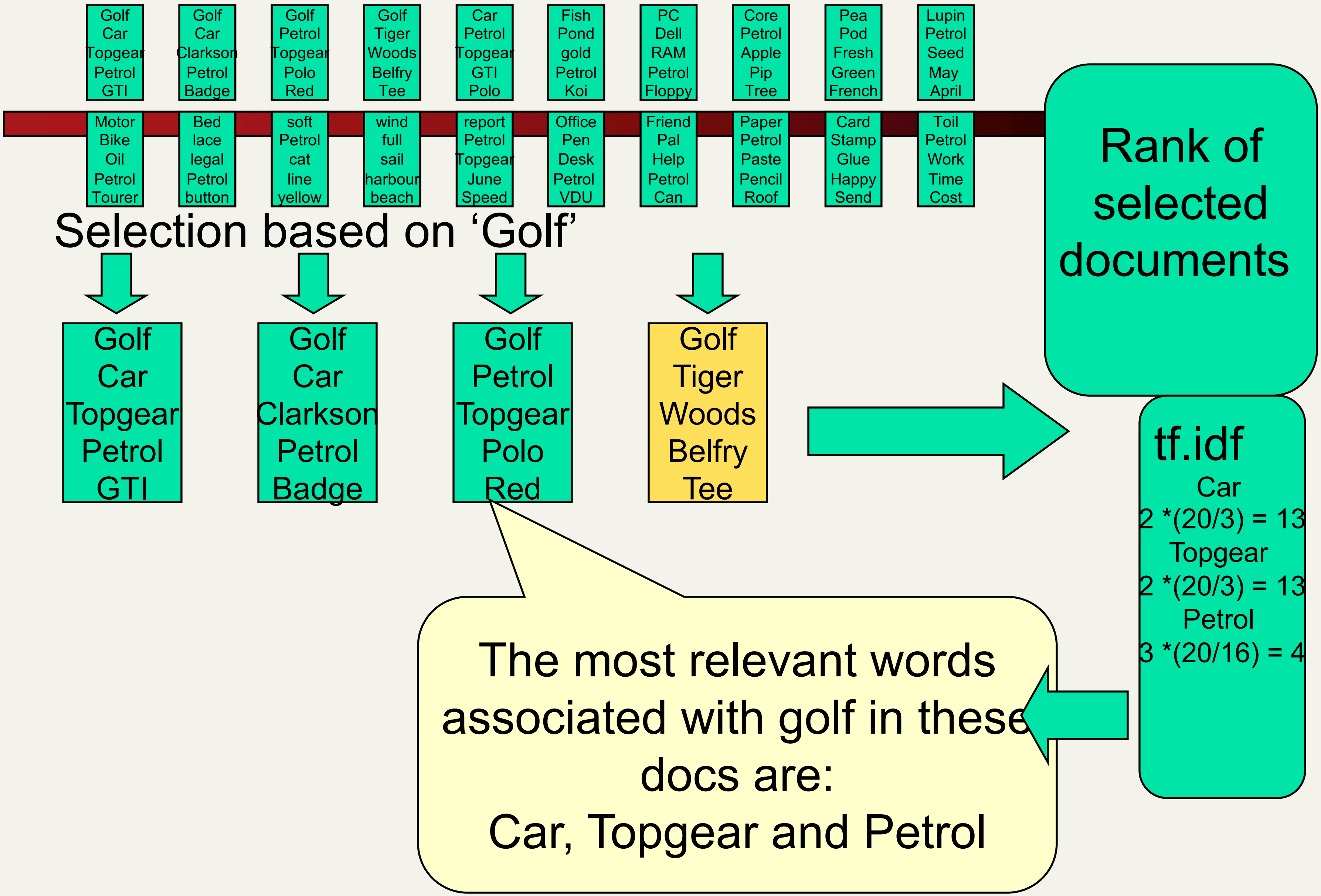
Document base (20)



Selection based on ‘Golf’



20 documents



20 docs

Golf Car Topgear Petrol GTI

Golf Car Clarkson Petrol Badge

Golf Petrol Topgear Polo Red

Golf Tiger Woods Belfry Tee

Car Petrol Topgear GTI Polo

Fish Pond gold Petrol Koi

PC Dell RAM Petrol Floppy

Core Petrol Apple Pip Tree

Pea Pod Fresh Green French

Lupin Petrol Seed May April

Motor Bike Oil Petrol Tourer

Bed lace legal Petrol button

soft Petrol cat line yellow

wind full sail harbour beach

report Petrol Topgear June Speed

Office Pen Desk Petrol VDU

Friend Pal Help Petrol Can

Paper Petrol Paste Pencil Roof

Card Stamp Glue Happy Send

Toil Petrol Work Time Cost

Selezione basata su 'Golf'

Golf Car Topgear Petrol GTI

Golf Car Clarkson Petrol Badge

Golf Petrol Topgear Polo Red

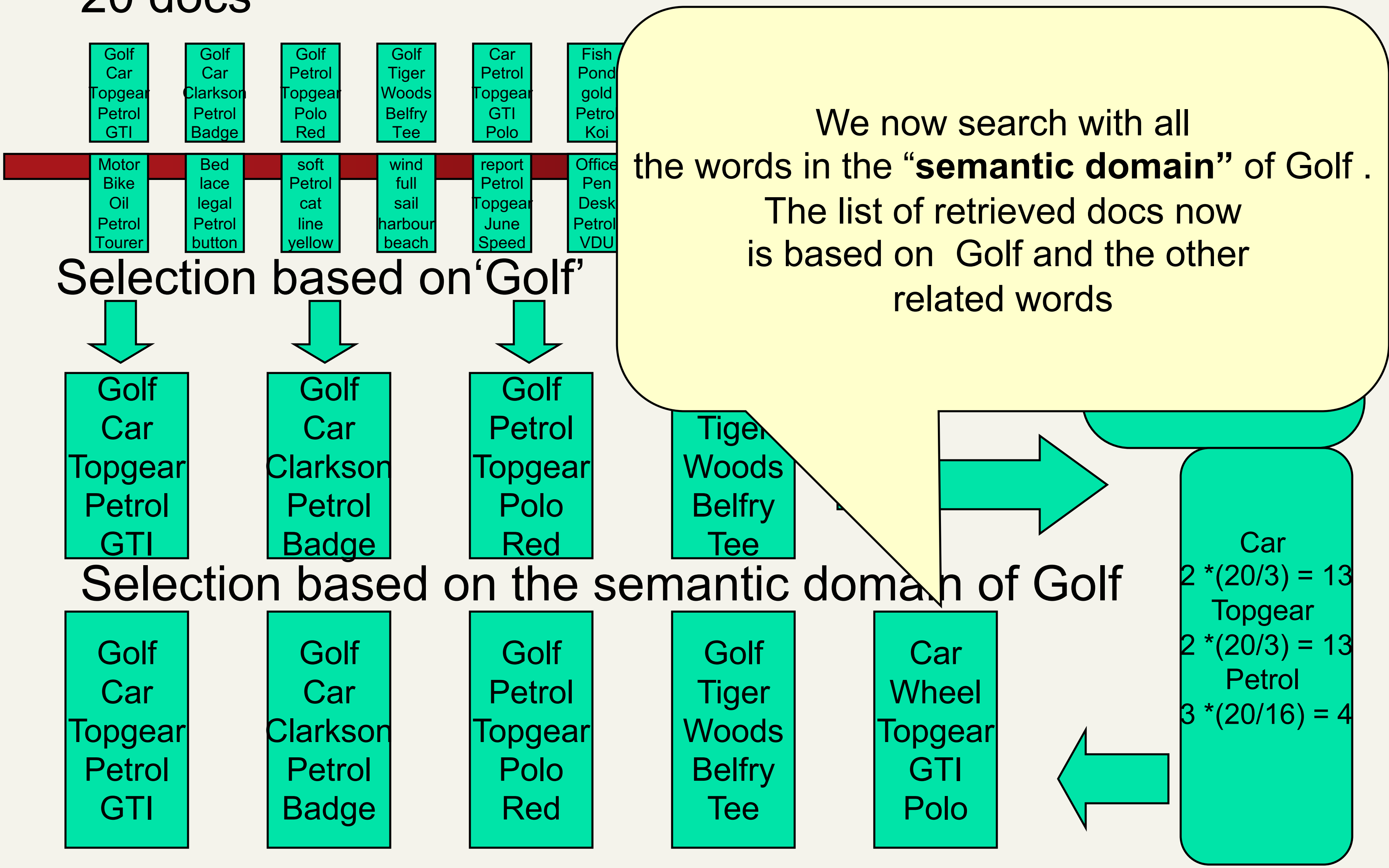
Golf Tiger Woods Belfry Tee

Rank of selected docs

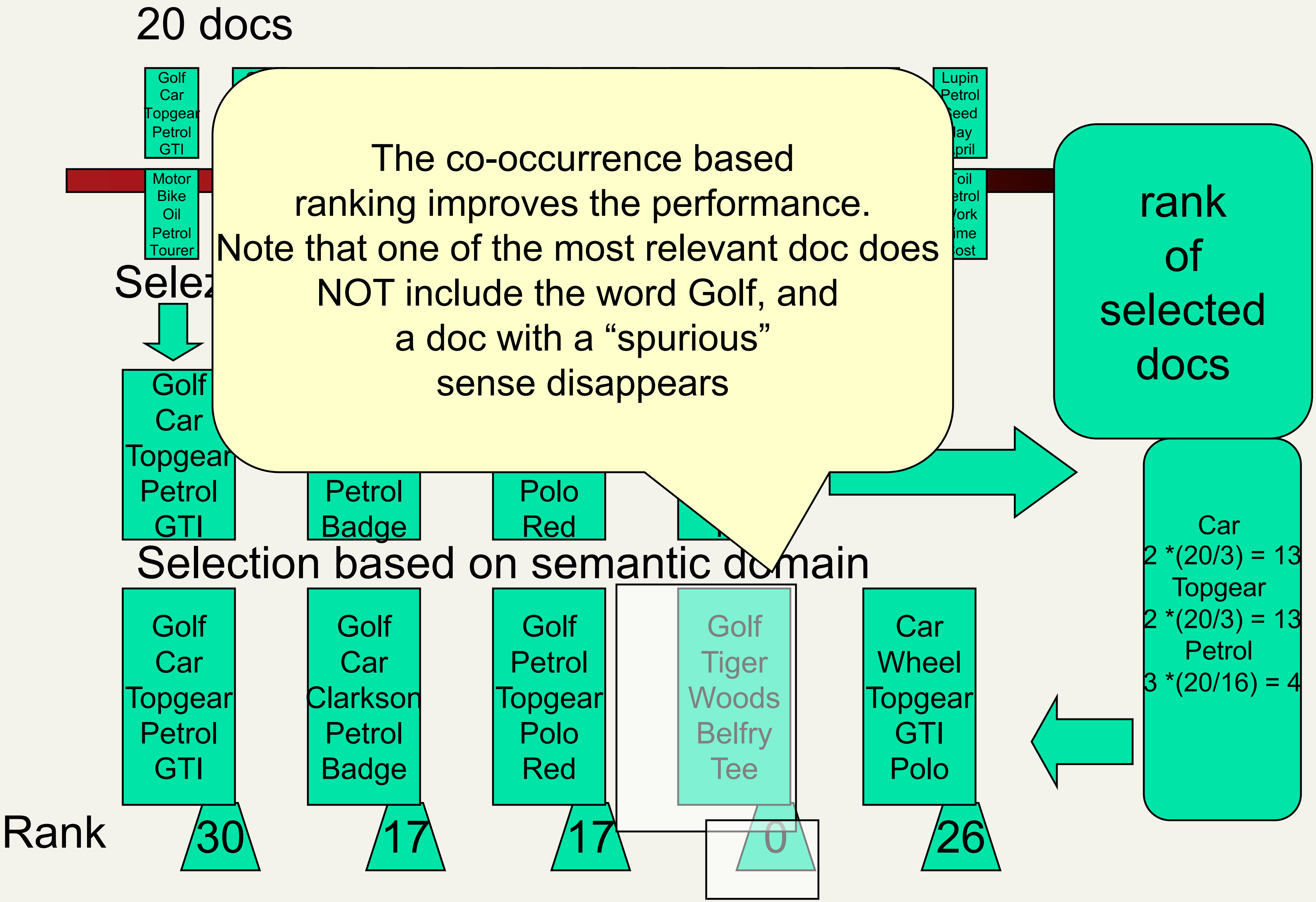
Car
2 *(20/3) = 13
Topgear
2 *(20/3) = 13
Petrol
3 *(20/16) = 4

If we consider the co-occurring terms with higher tf*idf, car e topgear turn out to be related to Golf more than petrol, wood,…

20 docs



Golf Car Topgear Petrol GTI
Golf Car Clarkson Petrol Badge
Golf Petrol Topgear Polo Red
Golf Tiger Woods Belfry Tee
Car Petrol Topgear GTI Polo
Fish Pond gold Petrol Koi
Motor Bike Oil Petrol Tourer
Bed lace legal Petrol button
soft Petrol cat line yellow
wind full sail harbour beach
report Petrol Topgear June Speed
Office Pen Desk Petrol VDU
Selection based on 'Golf'
Golf Car Topgear Petrol GTI
Golf Car Clarkson Petrol Badge
Golf Petrol Topgear Polo Red
Tiger Woods Belfry Tee
We now search with all the words in the "**semantic domain**" of Golf . The list of retrieved docs now is based on Golf and the other related words
Car
2 *(20/3) = 13
Topgear
2 *(20/3) = 13
Petrol
3 *(20/16) = 4
Selection based on the semantic domain of Golf
Golf Car Topgear Petrol GTI
Golf Car Clarkson Petrol Badge
Golf Petrol Topgear Polo Red
Golf Tiger Woods Belfry Tee
Car Wheel Topgear GTI Polo


20 docs
Golf Car Topgear Petrol GTI
Motor Bike Oil Petrol Tourer
Lupin Petrol
Selez
The co-occurrence based ranking improves the performance. Note that one of the most relevant doc does NOT include the word Golf, and a doc with a "spurious" sense disappears
rank of selected docs
Golf Car Topgear Petrol GTI
Petrol Badge
Polo Red
Car 2 *(20/3) = 13
Topgear 2 *(20/3) = 13
Petrol 3 *(20/16) = 4
Selection based on semantic domain
Golf Car Topgear Petrol GTI
Golf Car Clarkson Petrol Badge
Golf Petrol Topgear Polo Red
Golf Tiger Woods Belfry Tee
Car Wheel Topgear GTI Polo
Rank
30
17
17
0
26

Ranking with latent Semantic Indexing

- Previous example just gives the intuition
- Latent Semantic Indexing is an **algebraic** method to identify clusters of co-occurring terms, called “latent topics”, and to compute query-doc similarity in a **latent space**, in which every coordinate is a latent topic.
- A “latent” quantity is one which **cannot directly observed**, what is observed is a measurement which may include some amount of random errors (topics are “latent” in this sense: we observe them, but they are an approximation of “true” semantic topics)
- Since it is an algebraic method, needs some linear algebra background

The LSI method: how to detect "topics"

Linear Algebra Background

Eigenvalues & Eigenvectors

- **Eigenvectors** (for a square $m \times m$ matrix $\mathbf{S}$)

$$\mathbf{A}\underline{\mathbf{v}} = \lambda\underline{\mathbf{v}}$$

(right) eigenvector $\quad$ eigenvalue

$$\mathbf{v} \in \mathbb{R}^m \neq \mathbf{0} \qquad \lambda \in \mathbb{R}$$

Example

$$\begin{pmatrix} 6 & -2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\mathbf{A}\underline{\mathbf{v}} = \lambda\underline{\mathbf{v}}$$

- Def: A vector $\underline{v} \in R^n$, $\underline{v} \neq \underline{0}$, is an **eigenvector** of a matrix *mxm* A with corresponding **eigenvalue** λ, if:

  $\mathbf{A}\underline{\mathbf{v}} = \lambda\underline{\mathbf{v}}$

Algebraic method

- How many eigenvalues are there at most?

$$\mathbf{A}\underline{\mathbf{v}} = \lambda\underline{\mathbf{v}}$$

equation has a non-zero solution if $|A - \lambda\mathbf{I}| = 0$

$$A\mathbf{v} = \lambda\mathbf{v} \iff (A - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}$$

Where I is the identity matrix, and A is *mxm*
this is a m-th order equation in λ which can have **at most m distinct solutions** (roots of the characteristic polynomial) – <u>can be complex even though **A** is real.</u>

Example of eigenvector/eigenvalues

$$A = \begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \lambda = 4$$

$$A\underline{v} = \lambda \underline{v}$$

$$\begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 4\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 + -3(-1) \\ 3 + 5(-3) \end{pmatrix} = \begin{pmatrix} 4 \\ -12 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ -12 \end{pmatrix} = \begin{pmatrix} 4 \\ -12 \end{pmatrix}$$

We show that ***v*** is an eigenvector for A

Example of computation

remember

$$\det M = |M| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A = \begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\left| \begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix} - \lambda \ldots \right|$$

$$\left| \begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 1-\lambda & -1 \\ 3 & 5-\lambda \end{pmatrix} \right| = 0$$

$$\ldots 1)(\lambda - 5) + 3 = 0$$

$$\ldots - 6\lambda + 5 + 3 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 4)(\ldots 2) = 0$$

2 and 4 are the eigenvalues of S

Characteristic polynomial

$$(A - \lambda I)\underline{v} = \underline{0}$$

$$\begin{pmatrix} 1-4 & -1 \\ 3 & 5-4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -3\alpha - \beta = 0 \\ 3\alpha + \beta = 0 \end{cases} \qquad \beta = -3\alpha$$

$$(A - \lambda I)\underline{v} = \underline{0}$$

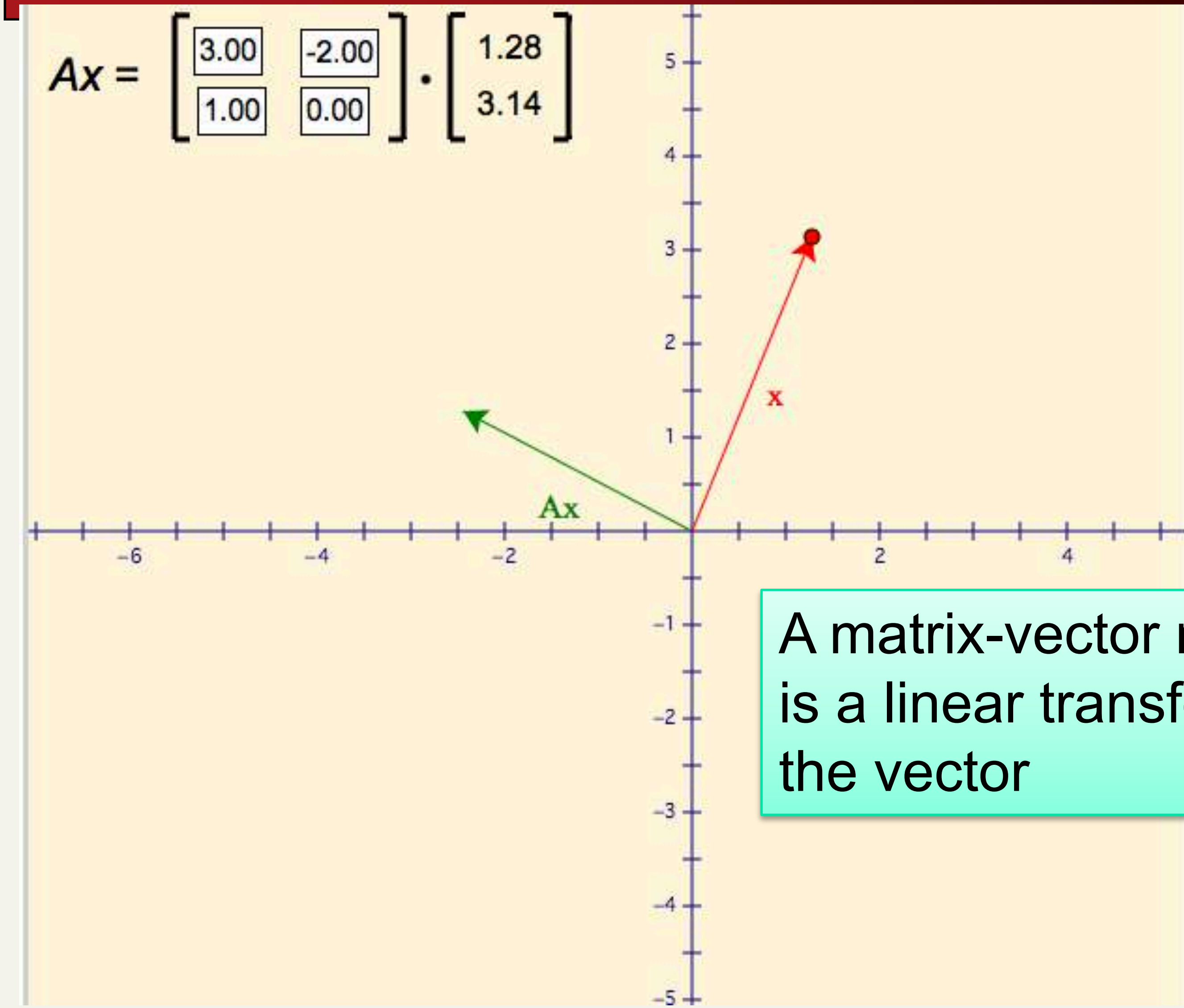
$$\begin{pmatrix} 1-2 & -1 \\ 3 & 5-2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -\alpha - \beta = 0 \\ 3\alpha + 3\beta = 0 \end{cases} \qquad \alpha = -\beta$$

Note that we compute only the **DIRECTION** of eigenvectors

Geometric interpretation



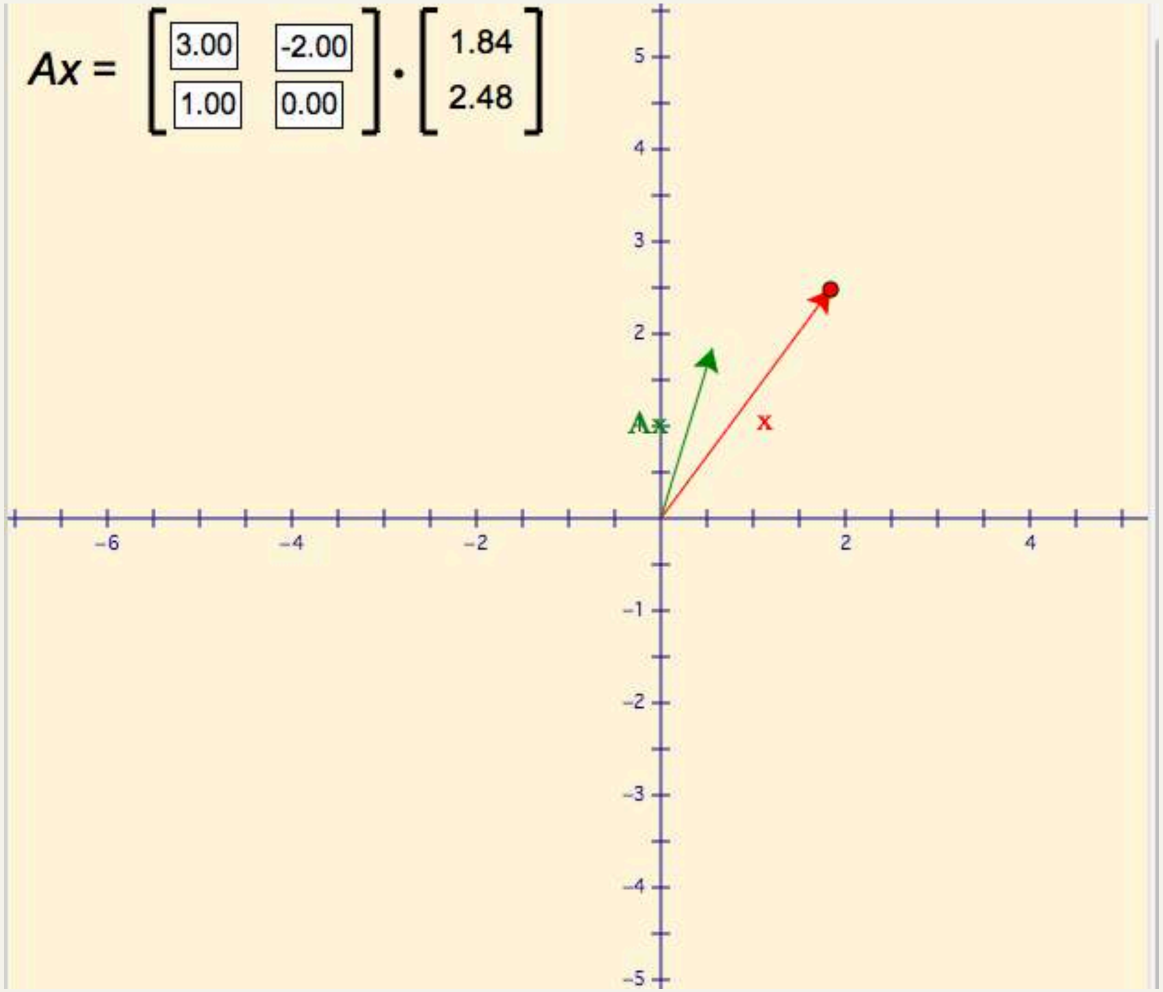
A matrix-vector multiplication Ax is a linear transformation over the vector

Matrix vector multiplication

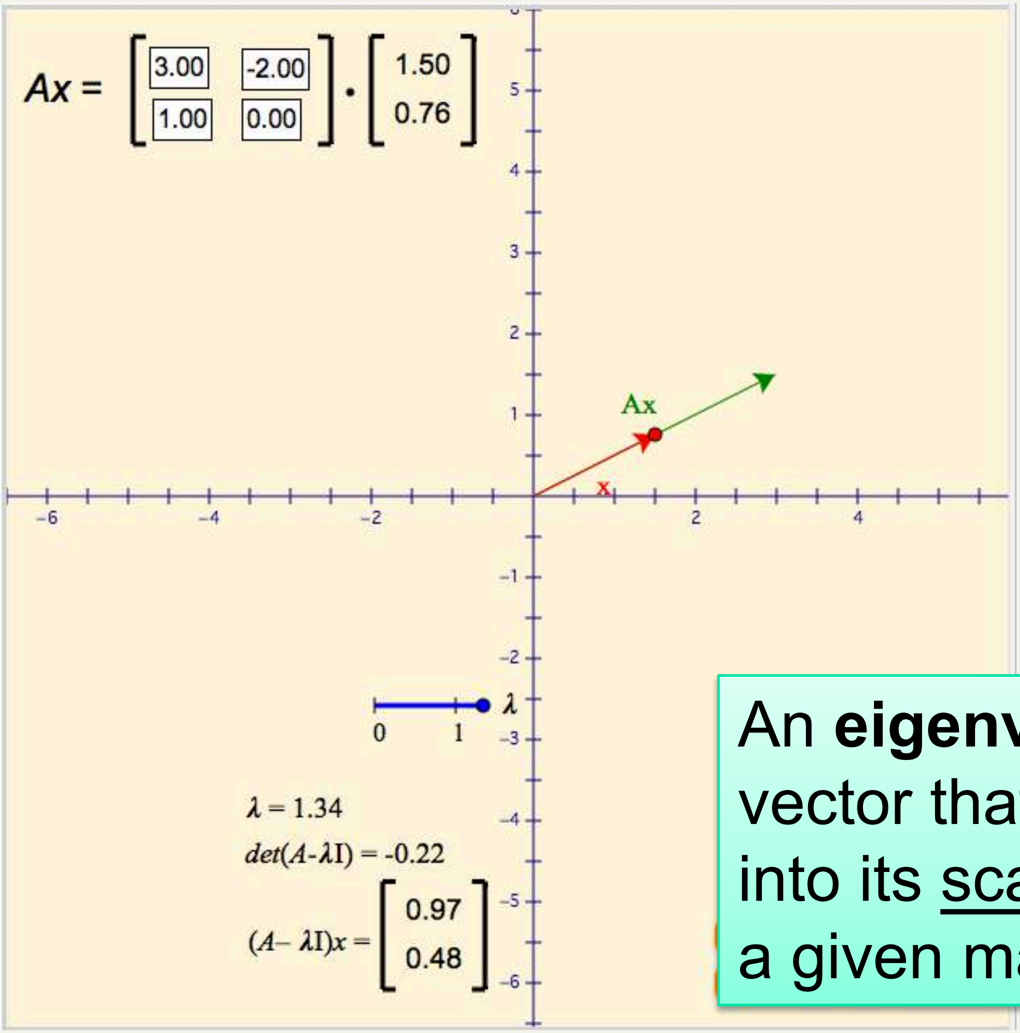
$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}.$$

Matrix multiplication by a vector = a linear transformation of the initial vector, that implies **rotation** and **translation of the original vector**

Geometric interpretation



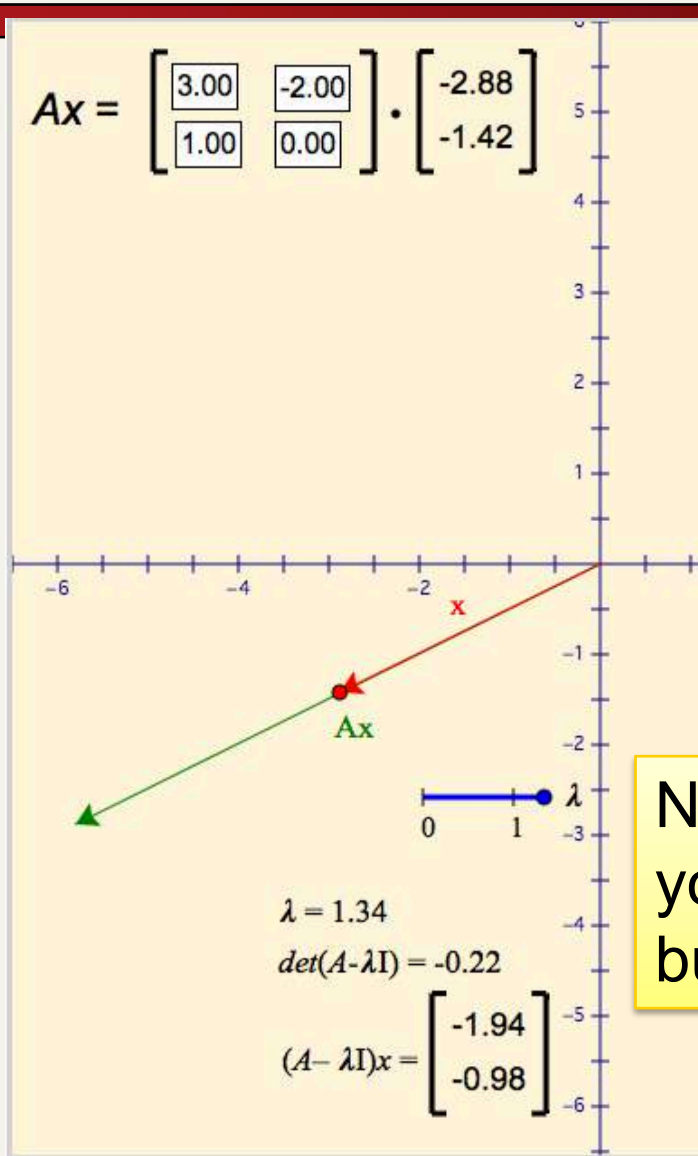
Geometric interpretation



$Ax=1.34x=\lambda x$

An **eigenvector** is a special vector that is transformed into its <u>scalar multiple</u> under a given matrix (<u>no rotation</u>!)

Geometric interpretation



Here we found another eigenvector for the matrix A

Note that for **any single eigenvalue** you have **infinite** eigenvectors, but they have **the same direction**

Matrix-vector multiplication

- Eigenvectors of different eigenvalues are **linearly independent** (i.e. $\forall \alpha_1 .. \alpha_n \rightarrow \alpha_1 v_1 + .. \alpha_n v_n \neq 0$)
- For **square normal** matrixes eigenvectors of different eigenvalues define an orthonormal space and they are othogonal.
- A square matrix is NORMAL iff it commutes with its transpose, i.e. $AA^T=A^TA$
- Example:
- $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow AA^T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = A^TA$

Difference between orthonormal and orthogonal?

- Orthogonal mean that the dot product is null (the cosin of the angle is zero).
  Orthonormal mean that the dot product is null **and** the norm of the vectors is equal to 1.
  What we are actually saying is that eigenvectors define a set of DIRECTIONS wich are orthogonal (=an othonormal space).
- If two or more vectors are orthonormal they are also orthogonal but the inverse is not true.

Why eigenvectors are orthonormal (if A is symmetric square matrix)

Let $\mathbf{v}_1$, $\mathbf{v}_2$ be two eigenvectors, and let λ_1 be the eigenvalue of $\mathbf{v}_1$, then we have:

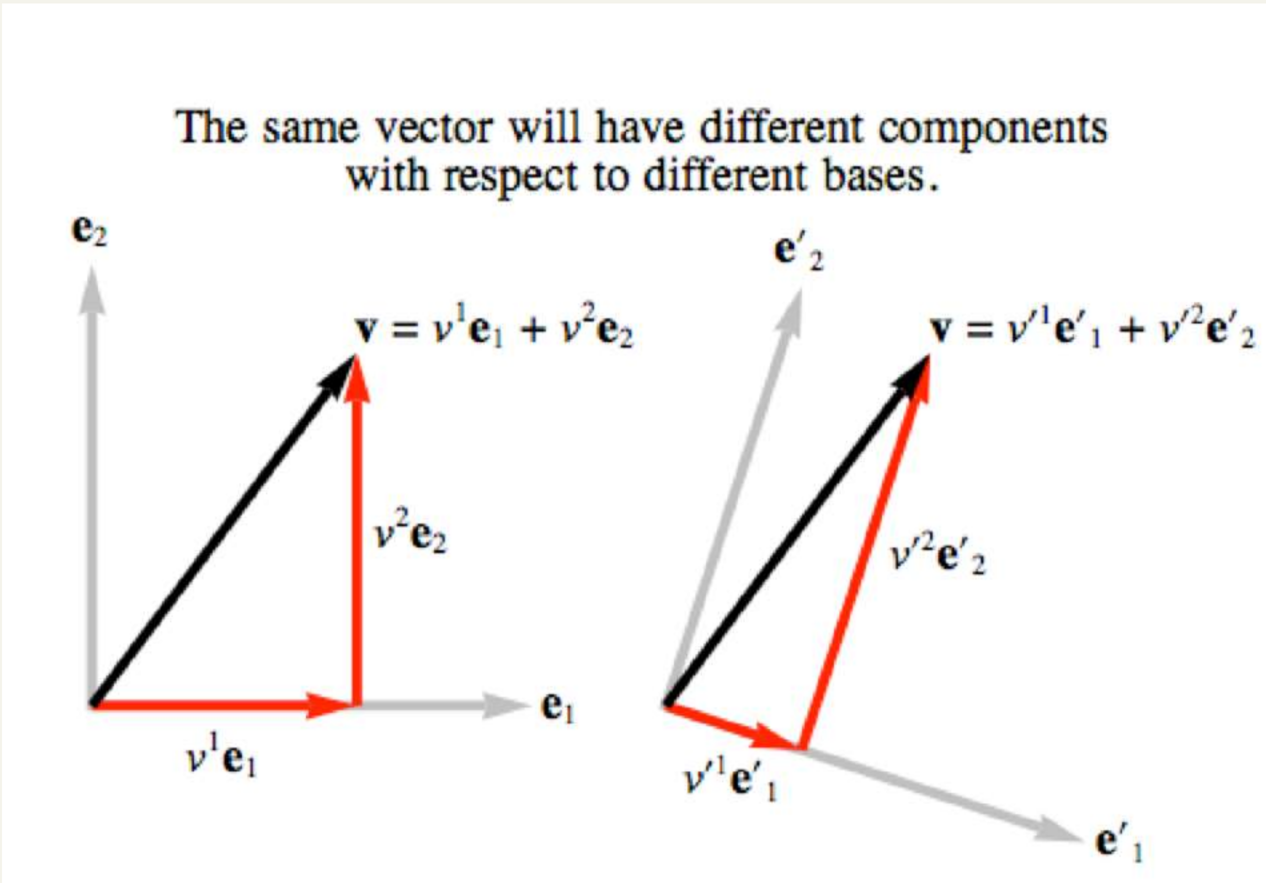
$$\lambda_1(v_2 \cdot v_1) = \lambda_1(v_2^T v_1) = (v_2^T \lambda_1 v_1) = (v_2^T A) v_1 = (A^T v_2)^T v_1$$

$$= (A v_2)^T v_1 = \lambda_2 v_2^T v_1 = \lambda_2 (v_2 \cdot v_1)$$

$$\Rightarrow (v_2 \cdot v_1) = 0 \; and \; v_2 \perp v_1$$

Either $\lambda_1 = \lambda_2$ or $(v_1 . v_2)$=0!

Example: projecting a vector on 2 orthonormal spaces (or "bases")



$\mathbf{e}_1,\mathbf{e}'_1$, $\mathbf{e}_2,\mathbf{e}'_2$ are **unary** vectors and v_1,v'_1, v_2,v'_2 are the coordinates of **v** along the directions of $\mathbf{e}_1,\mathbf{e}'_1$, $\mathbf{e}_2,\mathbf{e}'_2$

The effect of a matrix-vector multiplication is governed by eigenvectors and eigenvalues

Let $\vec{x}$ be a generic vector and A a normal matrix

- $A \cdot \vec{x} = A(x_1\overrightarrow{e_1} + x_2\overrightarrow{e_2} + x_3\overrightarrow{e_3})$ where x_i are the vector coordinates in the base defined by unary vectors $\overrightarrow{e_i}$
- Let's now project the very same vector $\vec{x}$ on the base defined by 3 eigenvectors of matrix *A*:

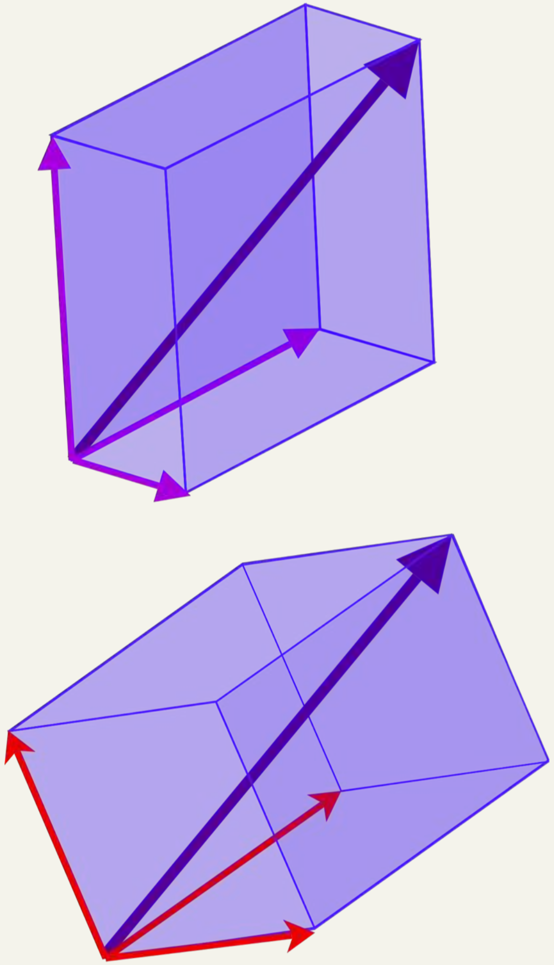
$$\vec{x} = x'_1\overrightarrow{e'_1} + x'_2\overrightarrow{e'_2} + x'_3\overrightarrow{e'_3} = \frac{x'_1}{|\overrightarrow{v_1}|}\overrightarrow{v_1} + \frac{x'_2}{|\overrightarrow{v_2}|}\overrightarrow{v_2} + \frac{x'_3}{|\overrightarrow{v_3}|}\overrightarrow{v_3}$$

- We then have: $x_1 = \frac{x'_1}{|\overrightarrow{v_1}|}$, $x_2 = \frac{x'_2}{|\overrightarrow{v_2}|}$, $x_3 = \frac{x'_3}{|\overrightarrow{v_3}|}$
- $A(x_1\overrightarrow{e_1} + x_2\overrightarrow{e_2} + x_3\overrightarrow{e_3}) = A(x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + x_3\overrightarrow{v_3})$
- $A(x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + x_3\overrightarrow{v_3}) = x_1\lambda_1\overrightarrow{v_1} + x_2\lambda_2\overrightarrow{v_2} + x_3\lambda_3\overrightarrow{v_3}$

The effect of a matrix-vector multiplication is governed by eigenvectors and eigenvalues (2)

$$A \cdot \vec{x} = x_1 \lambda_1 \overrightarrow{v_1} + x_2 \lambda_2 \overrightarrow{v_2} + x_3 \lambda_3 \overrightarrow{v_3}$$

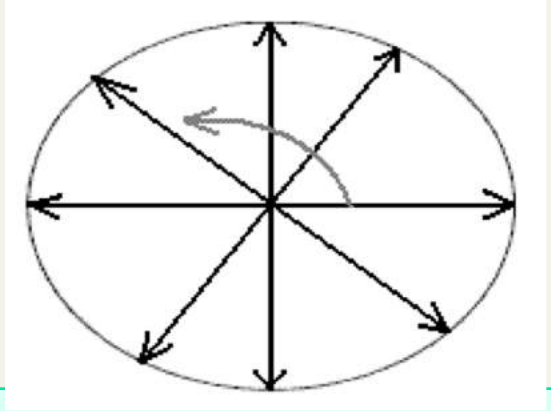
Even though *x* is an arbitrary vector, the action of *A* on *x* (transformation) is determined by the eigenvalues/vectors.

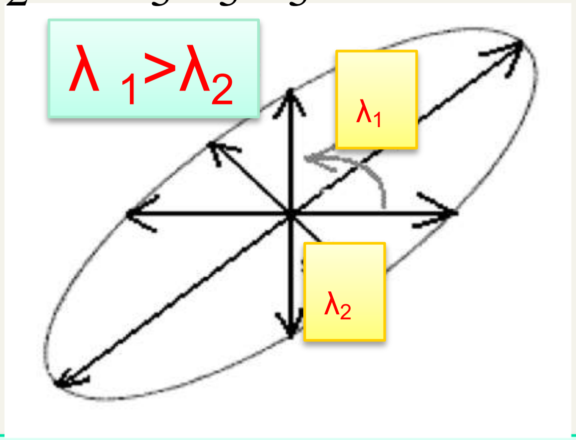


Geometric explanation: largest eigenvalues play the largest role in the "distortion" of the original vector

x is a generic vector with coordinates x_i; λ_i, v_i are the eigenvalues and eigenvectors of A

$$\boldsymbol{Ax} = \boldsymbol{x}_1 \lambda_1 \boldsymbol{v}_1 + \boldsymbol{x}_2 \lambda_2 \boldsymbol{v}_2 + \boldsymbol{x}_3 \lambda_3 \boldsymbol{v}_3$$

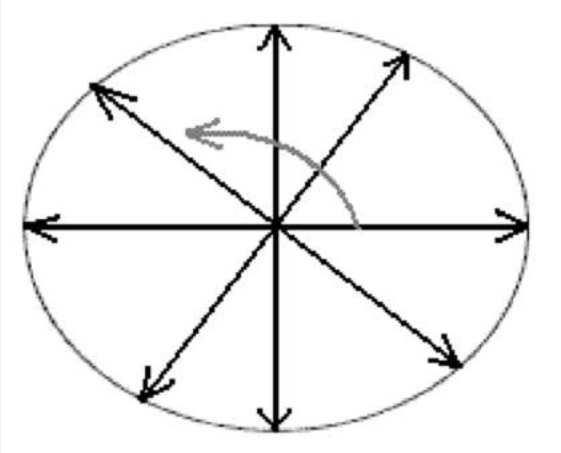


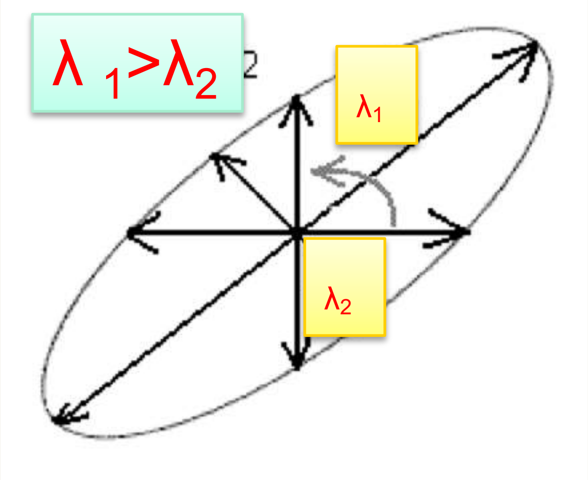


Multiplying a matrix and a vector has two effects over the vector: **rotation** (the coordinates of the vector change) and **scaling** (the length changes). The max compression and rotation **depends on the largest matrix's eigenvalues** λi

Geometric explanation

$$Ax = x_1\lambda_1 v_1 + x_2\lambda_2 v_2 + x_3\lambda_3 v_3$$





In the distortion, the max effect **is played by the biggest eigenvalues** (s1 and s2 in the picture)

The eigenvalues describe the distorsion operated by the matrix on the original vector

Summary so far

- A matrix A has eigenvectors v and eigenvalues λ, defined by $A\mathbf{v}=\lambda\mathbf{v}$
- Eigenvalues can be computed as:

$$A\mathbf{v} = \lambda\mathbf{v} \iff (A - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}$$

- We can compute only the the **direction** of eigenvectors, since for any eigenvalue there are infinite eigenvectors lying on the same direction
- If A is normal (i.e. if $AA^T=A^TA$) then the eigenvector form an othonormal basis
- The product of A by **ANY vector x** is a linear transformation of x where the **rotation** is determined by eigenvectors and the **translation** is determined by the eigenvalues. The biggest role in this transformation is played by the **biggest (principal) eigenvalues**.

Bad news..

More algebra..

Eigen/diagonal Decomposition

- Let A be a **square** matrix with ***m*** **orthogonal eigenvectors (hence, A is normal)**
- **Theorem**: Exists an **eigen decomposition**
  - $A=U\Lambda U^{-1}$
  - Λ is a diagonal matrix (all zero except the diagonal cells)

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m), \quad \lambda_i \geq \lambda_{i+1}$$

- Columns of ***U*** are **eigenvectors** of *A*
- Diagonal elements of Λ are **eigenvalues** of A

Diagonal decomposition: why/how

Let ***U*** have the eigenvectors as columns: $U = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$

Then, ***AU*** can be written

$$AU = A\begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \ldots & \lambda_n v_n \end{bmatrix} = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ldots & \\ & & \lambda_n \end{bmatrix}$$

Thus $\boldsymbol{AU=U\Lambda}$, or $\boldsymbol{U^{-1}AU=\Lambda}$

And $\boldsymbol{A=U\Lambda U^{-1}}$.

Example

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

$AU=U\Lambda$

$$A\,[\mathbf{v}_1\mathbf{v}_2] = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = [\mathbf{v}_1\mathbf{v}_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

From this we compute $\lambda_1=1$, $\lambda_2=3$

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 1\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$$

from which we get $v_{11}=-2v_{12}$

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 3\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$$

From which we get $v_{21}=0$ and v_{22} any real

Diagonal decomposition – example 2

Recall $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \qquad \lambda_1 = 1, \lambda_2 = 3.$

The eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ form $U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

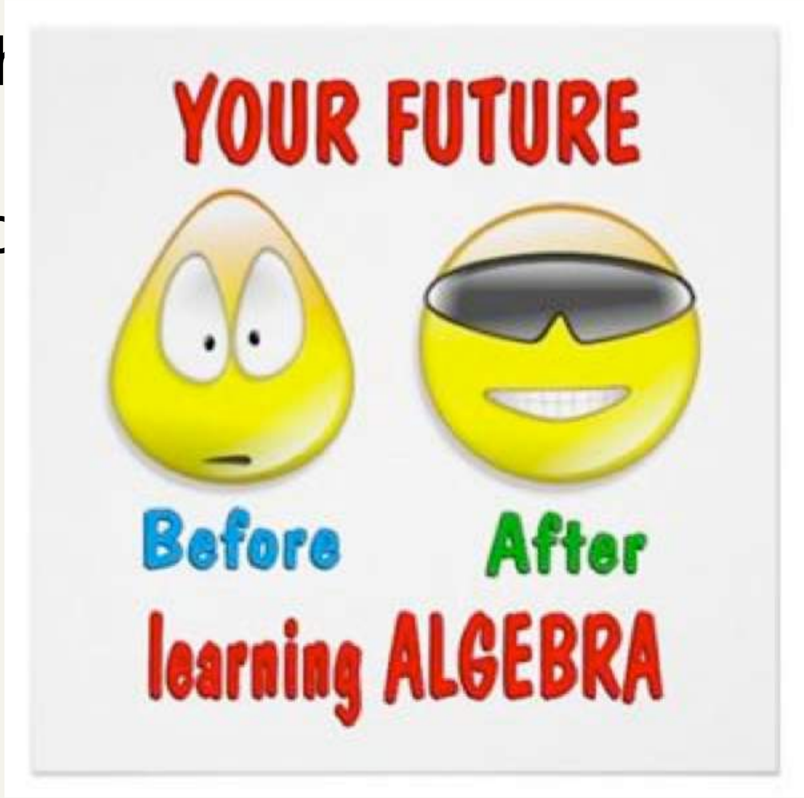
Inverting, we have $U^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Recall $UU^{-1} = 1$.

Then, $\boldsymbol{A=U\Lambda U^{-1}} =$ $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

So what?

- What do these matrices have to do with Information Retrieval and document ranking?
- Recall $M \times N$ term-document matrices …
- But everyth ormal matrices – and learn one last no



Singular Value Decomposition for non-square matrixes

For a **non-square** *real* $M \times N$ matrix $\mathbf{A}$ of rank r there exists a factorization (Singular Value Decomposition = **SVD**) as follows:

$$A = U\Sigma V^T$$

$M \times M$ $\quad$ $M \times N$ $\quad$ V is $N \times N$

The columns of ***U*** are the orthogonal eigenvectors of $\boldsymbol{AA^T}$ *(called* ***left singular vectors)***.

The columns of ***V*** (rows of $\mathbf{V^T}$) are the orthogonal eigenvectors of $\boldsymbol{A^TA}$ (called right singular eigenvector). NOTE THAT $\boldsymbol{AA^T}$ ***and*** $\boldsymbol{A^TA}$ are square symmetric (and hence **NORMAL**)

Eigenvalues $\lambda_1 \ldots \lambda_r$ of $\boldsymbol{AA^T}$ = eigenvalues of $\boldsymbol{A^TA}$ and: $\sigma_i = \sqrt{\lambda_i}$

$$\Sigma = diag(\sigma_1 \ldots \sigma_r)$$

Singular values of A

An example

Find the SVD of A, $U\Sigma V^T$, where $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$

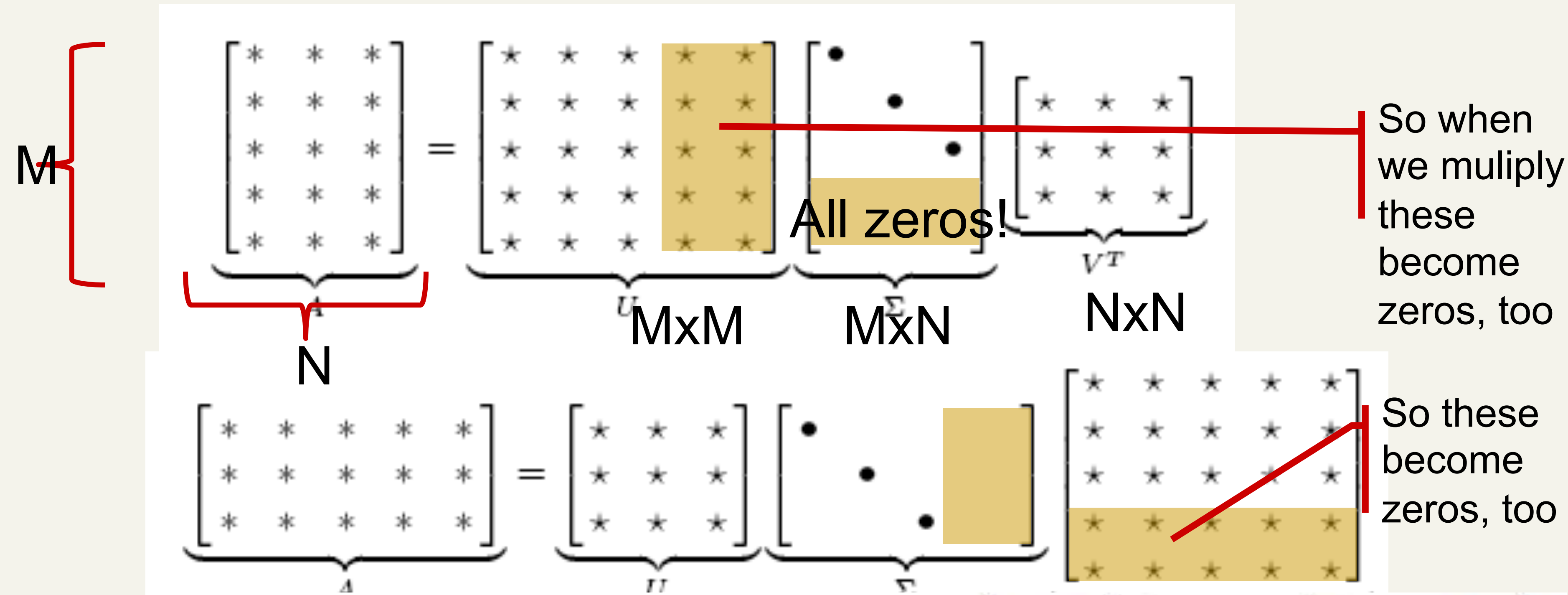
$$A = U\Sigma V^T = U \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{pmatrix}$$

$$A = U\Sigma V^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ -1/3 \end{pmatrix}$$

Singular Value Decomposition

- Illustration of SVD dimensions and sparseness



$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & \boxed{0} \\ 0 & 3 & \boxed{0} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ \boxed{2/3} & \boxed{-2/3} & \boxed{-1/3} \end{pmatrix}$$

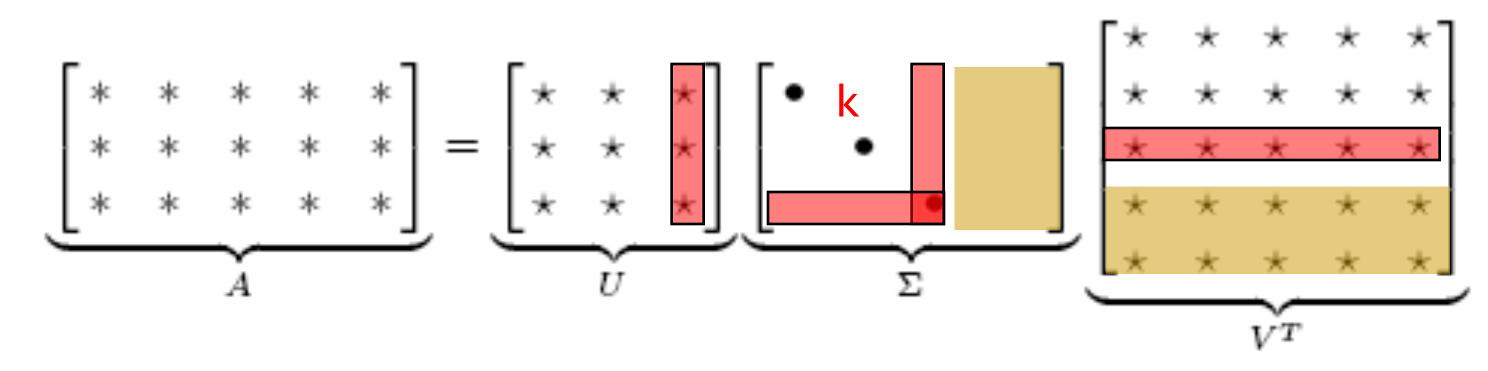
Back to matrix-vector multiplication

- Remember what we said? In a matrix vector multiplication the biggest role is played by the biggest eigenvalues
- The diagonal matrix Σ has the eigenvalues of A^TA (called the *singular values* $\boldsymbol{\sigma}$ of A) in decreasing order along the diagonal
- We can therefore apply an **approximation** by setting $\sigma_i=0$ if $\sigma_i \leq \theta$ and only consider only the first k singular values

Reduced SVD

- If we retain only k *highest* singular values, and set the rest to 0, then **we don't need the matrix parts in red**
- Then Σ is $k \times k$, U is $M \times k$, V^T is $k \times N$, and A_k is $M \times N$
- This is referred to as the ***reduced SVD, or rank k approximation***

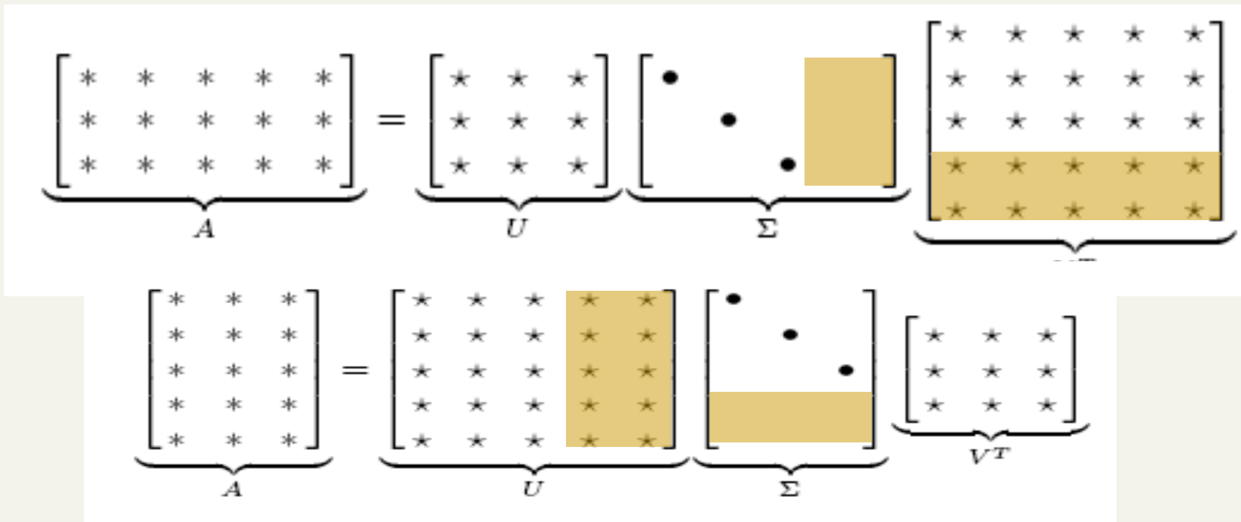
Now all the red and yellow parts are zeros!!



Let's recap

M<N

M>N



Since the yellow part is zero, an **exact** representation of A is:

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots + \sigma_r u_r v_r^T$$

$$r = \min(M, N)$$

But "for some" k<r, a **good approximation** is:

$$A_k = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots + \sigma_k u_k v_k^T$$

Example of rank approximation

$$\begin{bmatrix}1&0&0&0&2\\0&0&3&0&0\\0&0&0&1&0\\0&4&0&0&0\end{bmatrix} = \begin{bmatrix}0&0&1&\cancel{0}\\0&1&0&\cancel{0}\\0&0&0&\cancel{-1}\\1&0&0&\cancel{0}\end{bmatrix} \times \begin{bmatrix}4&0&0&0&0\\0&3&0&0&0\\0&0&\sqrt{5}&0&0\\0&0&0&1&0\end{bmatrix} \times \begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\\sqrt{0.2}&0&0&0&\sqrt{0.8}\\\cancel{0}&\cancel{0}&\cancel{0}&\cancel{1}&\cancel{0}\\\cancel{-\sqrt{0.8}}&\cancel{0}&\cancel{0}&\cancel{0}&\cancel{\sqrt{0.2}}\end{bmatrix}$$

A

$$[A]^* = \begin{bmatrix}001\\010\\000\\100\end{bmatrix} \times \begin{bmatrix}400\\030\\00\sqrt{5}\end{bmatrix} \times \begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\\sqrt{0.2}&0&0&0&\sqrt{0.8}\end{bmatrix}$$

A*=

0.981	0.000	0.000	0.000	1.985
0.000	0.000	3.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
0.000	4.000	0.000	0.000	0.000

≅A

Approximation error

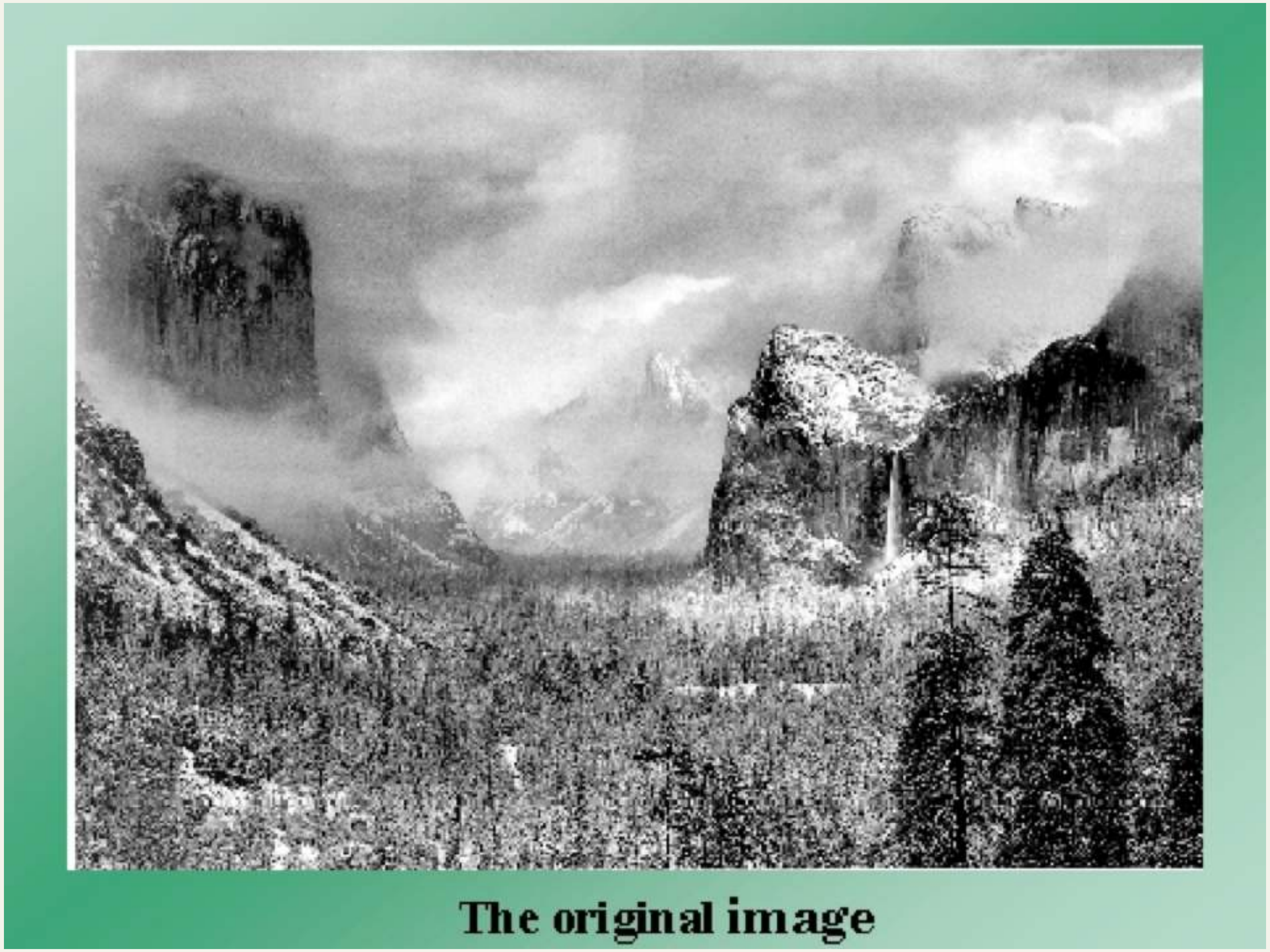
- How good (bad) is this approximation?
- It's the best possible, measured by the Frobenius norm of the error:

$$\min_{X:rank(X)=k} \|A - X\|_F = \|A - A_k\|_F = \sigma_{k+1} \qquad \sigma_i = \sqrt{\lambda_i}$$

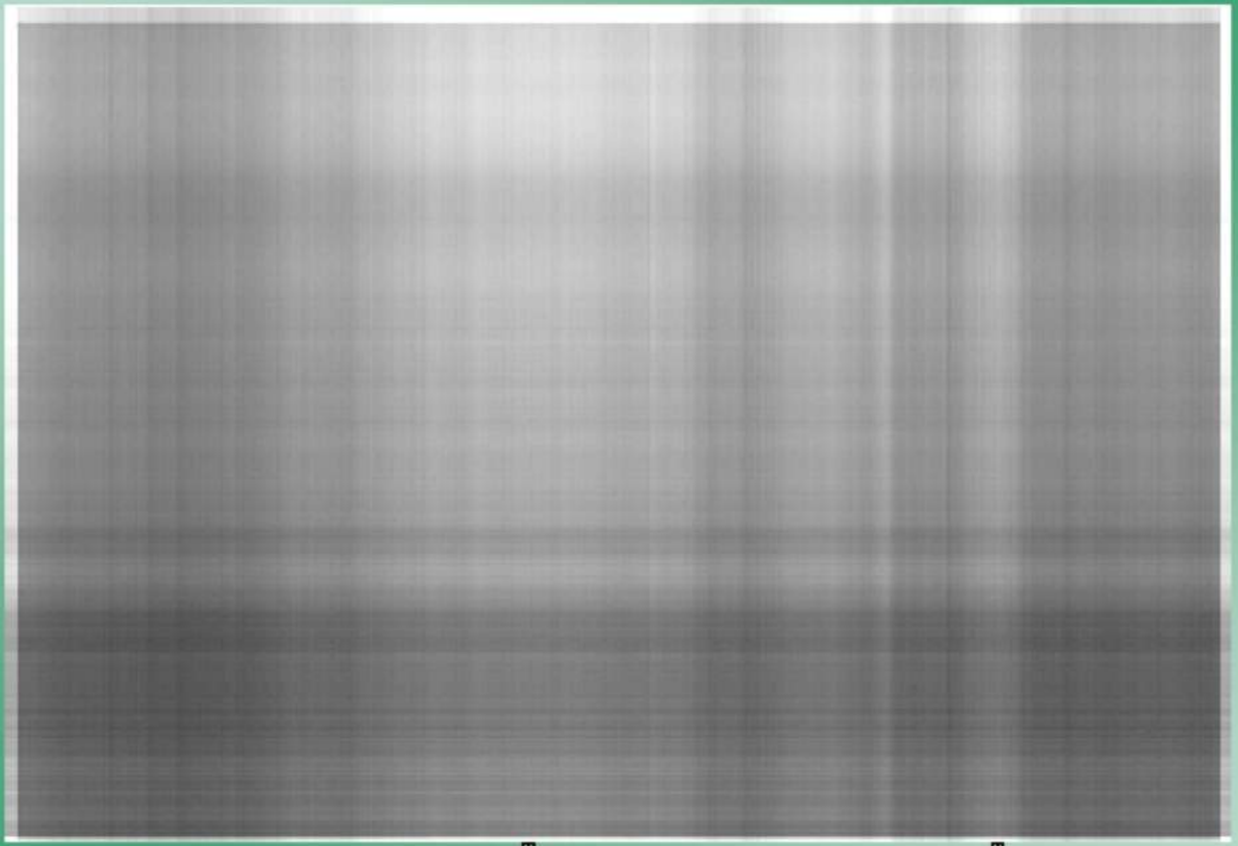
where the σ_i are ordered such that $\sigma_i \geq \sigma_{i+1}$.

  - Suggests why Frobenius error drops as *k* increases.

Images gives a better intuition (image = matrix of pixels)

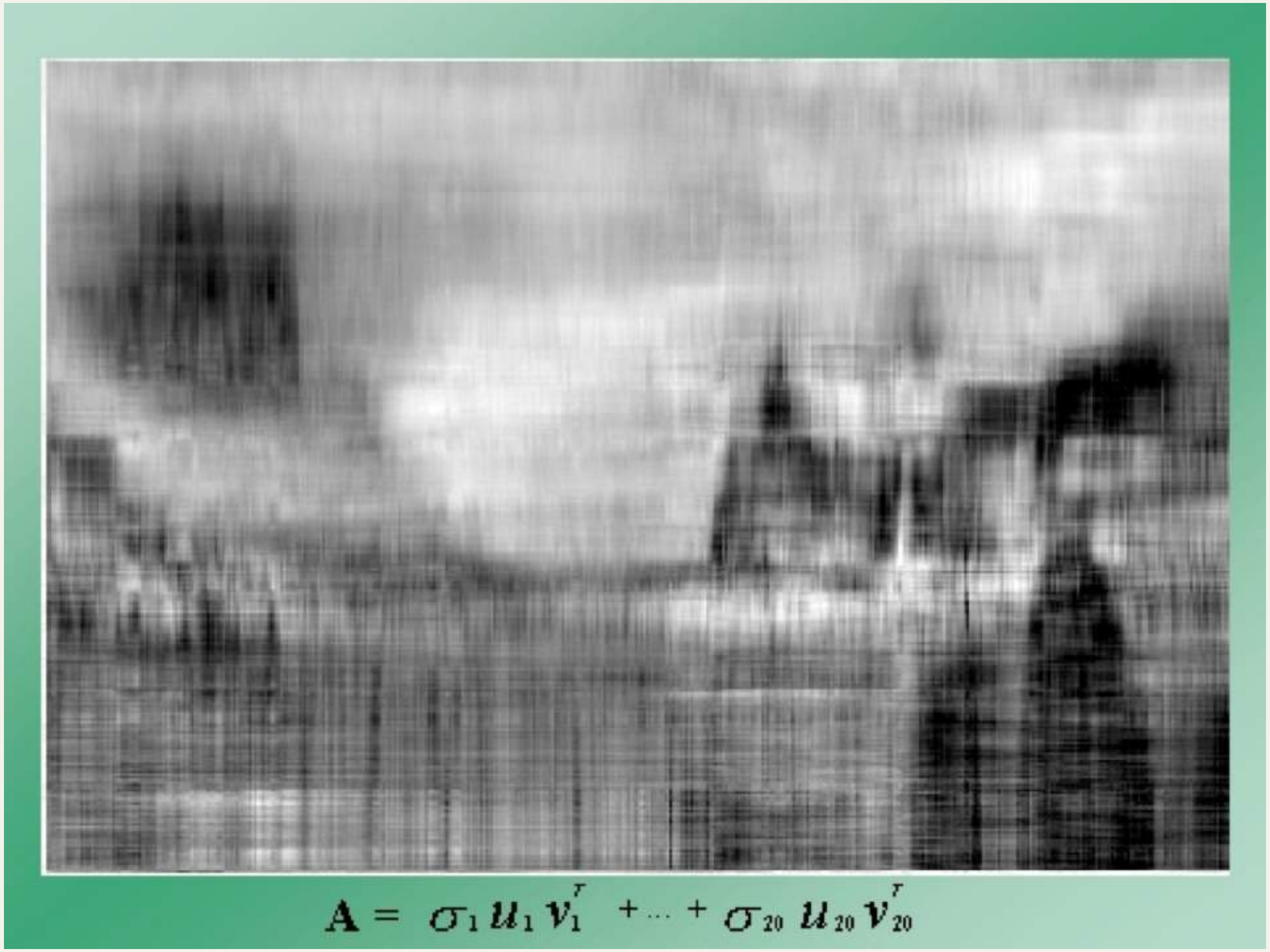


K=10

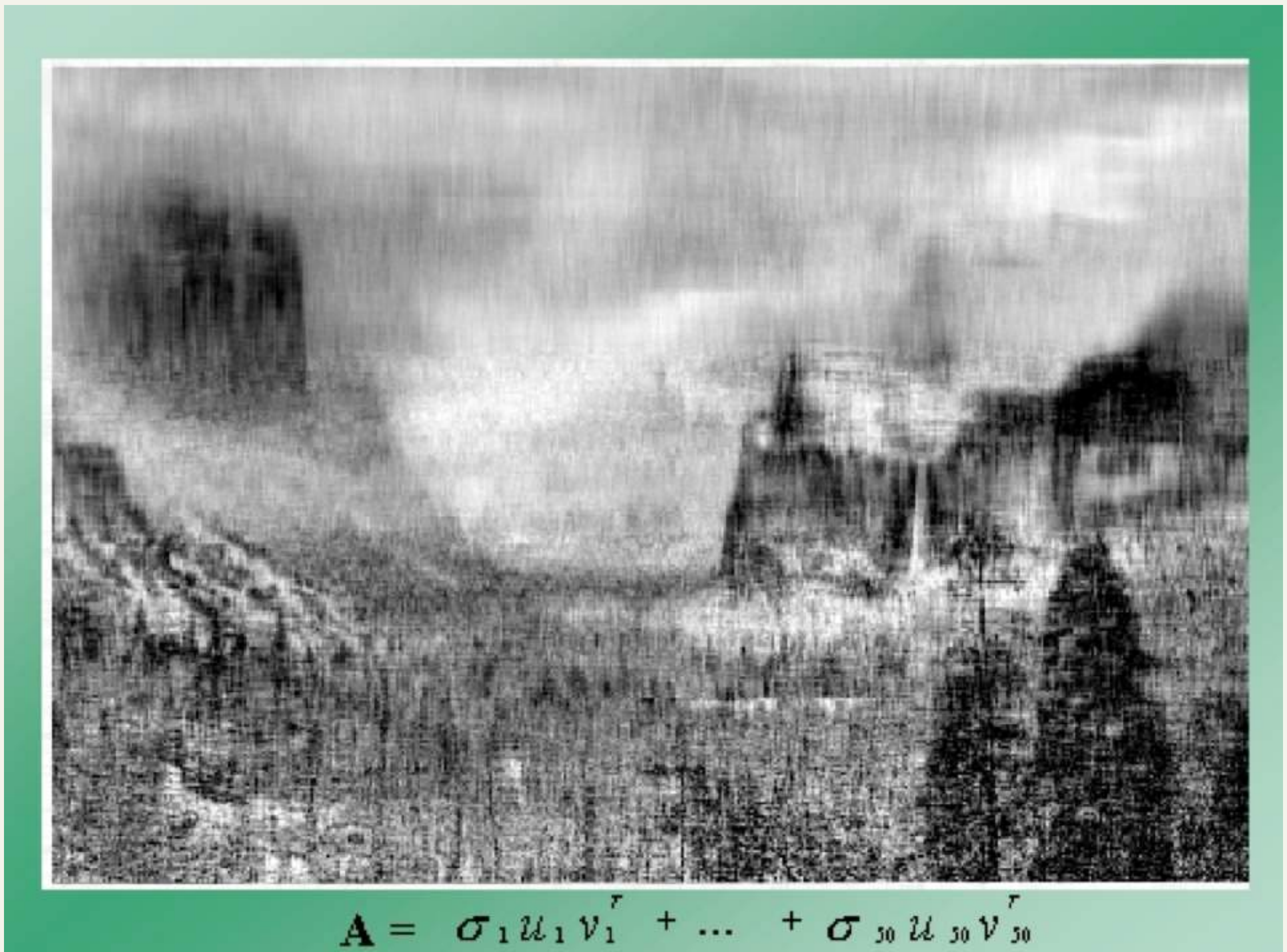


$$A = \sigma_1 u_1 v_1^T + \dots + \sigma_{10} u_{10} v_{10}^T$$

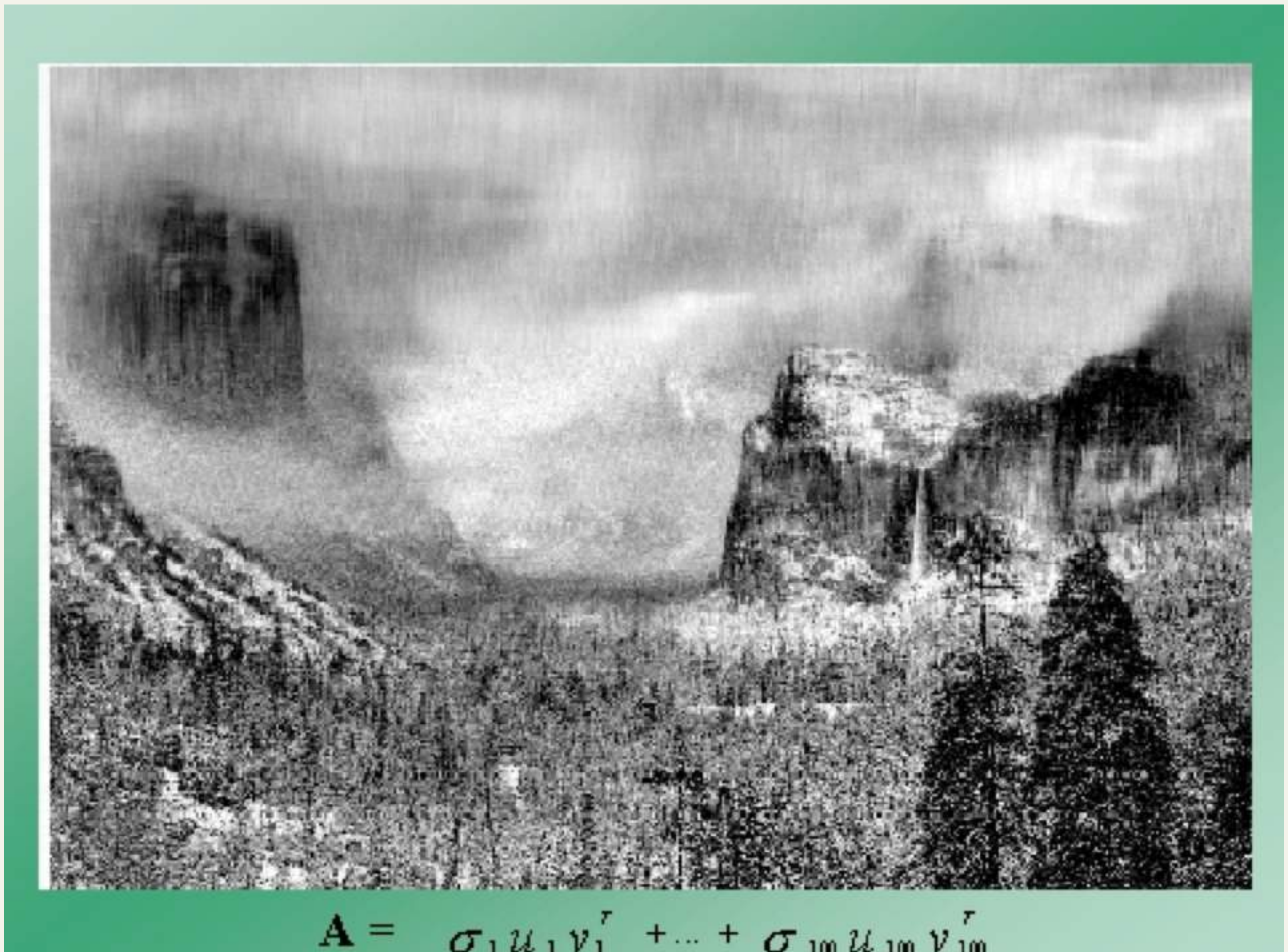
K=20



K=30



K=100



K=322 (the original image)


The original image

We obtained our approximation by summing up only the first 100 terms of the singular value decomposition.

This approximation reduced the amount of information in our image by nearly **70% !!!**

We save space!! But this is only one issue

So, finally, back to IR!!!

- Our initial problem was:
  - the term-document (MxN) matrix A is highly sparse (has many zeros)
  - However, since groups of terms tend to co-occur together, can we identify the **semantic space** of these clusters of terms, and apply the vector space model in the semantic space defined by such clusters (rather than the space of terms)?
- What we learned so far:
  - Matrix A can be decomposed, and rank-k approximated using SVD
  - Does this help solving our initial problems?

A is our term document matrix

- **Latent Semantic Indexing via the SVD**

$$A = U\Sigma V^T$$

$M \times M$ | $M \times N$ | V is $N \times N$

The columns of ***U*** are orthogonal eigenvectors of $\boldsymbol{AA^T}$.

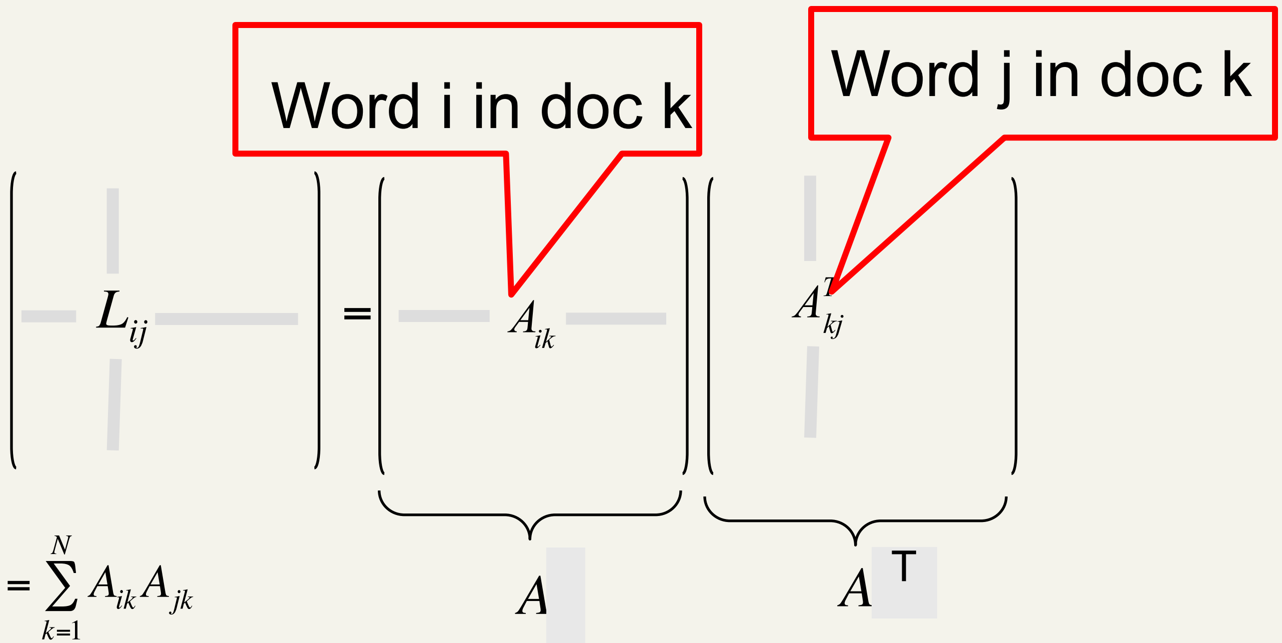
The columns of ***V*** are orthogonal eigenvectors of $\boldsymbol{A^TA}$.

Eigenvalues $\lambda_1 \ldots \lambda_r$ of $\boldsymbol{AA^T}$ are the eigenvalues of $\boldsymbol{A^TA}$.

- If A is a term/document matrix, then AA^T and A^T A are the (square) matrixes of **term** and **document** co-occurrences, repectively

Meaning of A^TA and AA^T

$L = A\ A^T =$



$$L_{ij} = \sum_{k=1}^{N} A_{ik} A^T_{kj} = \sum_{k=1}^{N} A_{ik} A_{jk}$$

L_{ij} depends on the number of documents d_k in which wi and wj co-occurr (the non-zero products $A_{ik}A^T_{kj}$ of the sum)

Similarly, L^T_{ij} depends on the number of common documents for two word pairs (or vice-versa if A is a document-term matrix rather than term-document)

Example

Example of text data: Titles of Some Technical Memos

c1: *Human* machine *interface* for ABC *computer* applications
c2: A *survey* of *user* opinion of *computer system response time*
c3: The *EPS user interface* management *system*
c4: *System* and *human system* engineering testing of *EPS*
c5: Relation of *user* perceived *response time* to error measurement

m1: The generation of random, binary, ordered *trees*
m2: The intersection *graph* of paths in *trees*
m3: *Graph minors* IV: Widths of *trees* and well-quasi-ordering
m4: *Graph minors*: A *survey*

Term-document matrix

A =

	c1	c2	c3	c4	c5	m1	m2	m3	m4
human	1	0	0	1	0	0	0	0	0
interface	1	0	1	0	0	0	0	0	0
computer	1	1	0	0	0	0	0	0	0
user	0	1	1	0	1	0	0	0	0
system	0	1	1	2	0	0	0	0	0
response	0	1	0	0	1	0	0	0	0
time	0	1	0	0	1	0	0	0	0
EPS	0	0	1	1	0	0	0	0	0
survey	0	1	0	0	0	0	0	0	1
trees	0	0	0	0	0	1	1	1	0
graph	0	0	0	0	0	0	1	1	1
minors	0	0	0	0	0	0	0	1	1

Term co-occurrences example

	c1	c2	c3	c4	c5	m1	m2	m3	m4
human	1	0	0	1	0	0	0	0	0
interface	1	0	1	0	0	0	0	0	0
computer	1	1	0	0	0	0	0	0	0
user	0	1	1	0	1	0	0	0	0
system	0	1	1	2	0	0	0	0	0
response	0	1	0	0	1	0	0	0	0
time	0	1	0	0	1	0	0	0	0
EPS	0	0	1	1	0	0	0	0	0
survey	0	1	0	0	0	0	0	0	1
trees	0	0	0	0	0	1	1	1	0
graph	0	0	0	0	0	0	1	1	1
minors	0	0	0	0	0	0	0	1	1

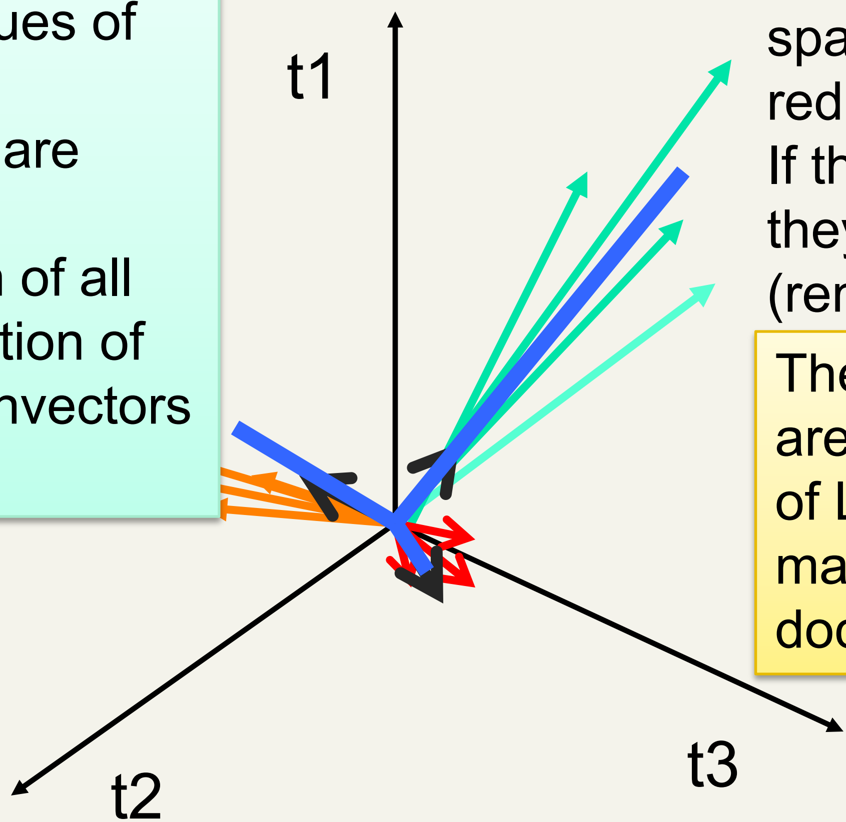
$$L_{\text{trees,graph}} = (000001110) \bullet (000000111)^T = 2$$

So the matrix $L=AA^T$ is the matrix of term co-occurrences in docs

- **Remember**: eigenvectors of a matrix define an orthonormal space
- **Remember**: bigger eigenvalues define the “main” directions of this space
- **But**: Matrixes L and L^T are SIMILARITY (co-occurrence) matrixes (respectively, of terms and of documents). They define a SIMILARITY SPACE (the orthonormal space of their eigenvectors)
- **If the matrix elements are word co-occurrences, bigger eigenvalues are associated to bigger groups of similar words**
- **Similarly, bigger eigenvalues of $L^T=A^TA$ are associated with bigger groups of similar documents (those in which co-occur the same terms)**

LSI: the intuition

The blue segments give the intuition of eigenvalues of $L^T = A^T A$
Bigger eigenvalues are those for
which the projection of all vectors on the direction of correspondent eigenvectors is higher



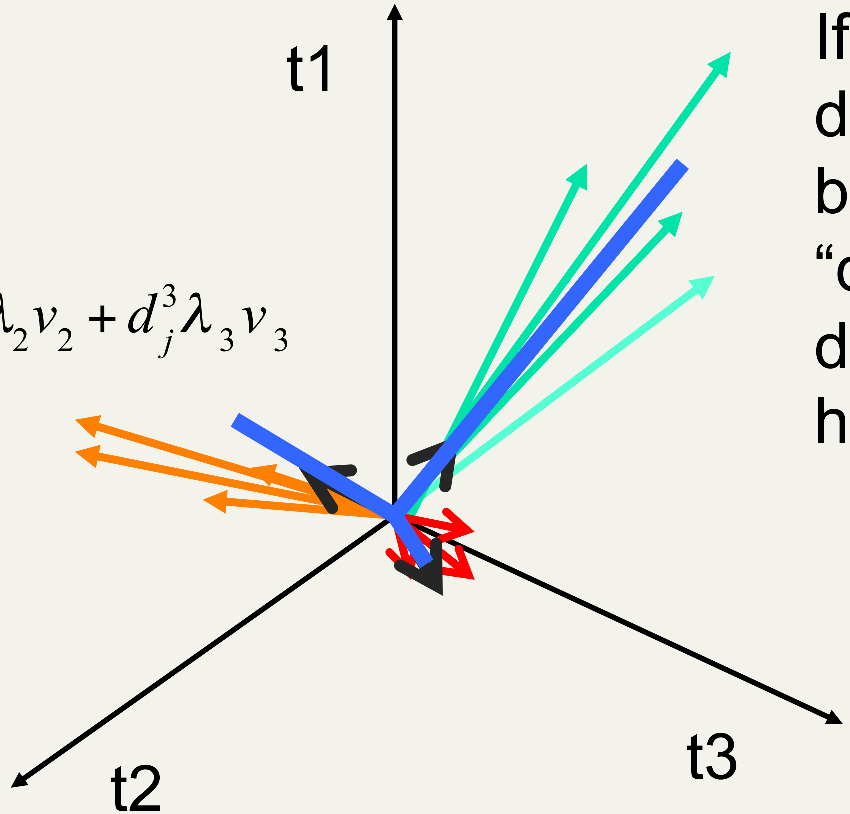
Projecting *A* in the term space: green, yellow and red vectors are documents.
If they form small angles, they have common words (remember cosin-sim)

The black vector are the unary eigenvector of L^T: they represent the main "directions" of the document vectors

LSI intuition

$$\vec{d}_j = (d_j^1, d_j^2, d_j^3)$$

$$L^T \vec{d}_j = d_j^1 \lambda_1 v_1 + d_j^2 \lambda_2 v_2 + d_j^3 \lambda_3 v_3$$

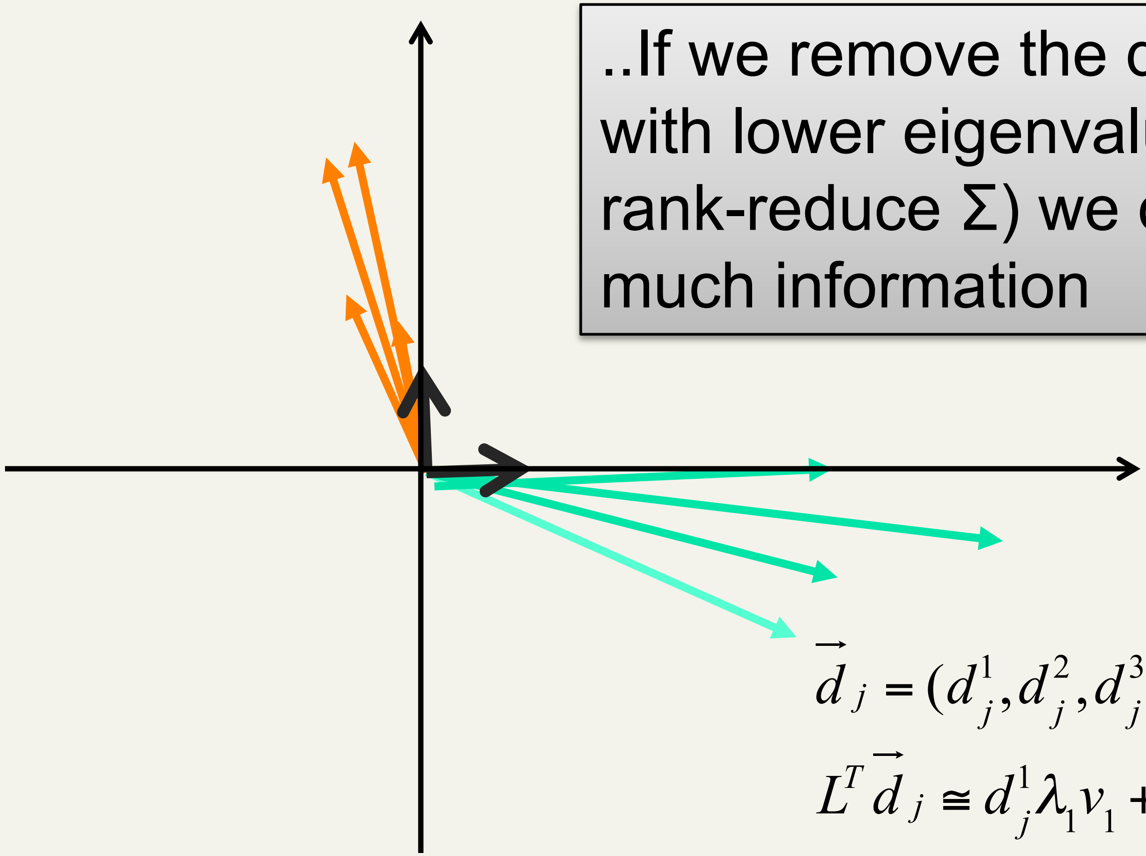


If we multiply all document vectors by $L^T = A^T A$, their “distorsion” is mostly determined by the highest eigenvalues

We now project our document vectors on the reference orthonormal system represented by the 3 black vectors

LSI intuition

..If we remove the dimension(s) with lower eigenvalues (i.e. if we rank-reduce Σ) we don't loose much information



$$\vec{d}_j = (d_j^1, d_j^2, d_j^3)$$

$$L^T \vec{d}_j \cong d_j^1 \lambda_1 v_1 + d_j^2 \lambda_2 v_2$$

Remember that the two "new" axis represent a combination of co-occurring words e.g. a **latent semantic space**

Example

	d1	d2	d3	d4
t1	1.000	1.000	0.000	0.000
t2	1.000	0.000	0.000	0.000
t3	0.000	0.000	1.000	1.000
t4	0.000	0.000	1.000	1.000

We project terms and docs on two dimensions, **v1** and **v2** (the principal eigenvectors)

$$\begin{bmatrix} 0.000 & -0.851 & -0.526 & 0.000 \\ 0.000 & -0.526 & 0.851 & 0.000 \\ -0.707 & 0.000 & 0.000 & -0.707 \\ -0.707 & 0.000 & 0.000 & 0.707 \end{bmatrix} \times \begin{bmatrix} 2.000 & 0.000 \\ 0.000 & 1.618 \end{bmatrix} \times \begin{bmatrix} 0.000 & 0.000 & -0.707 & -0.707 \\ -0.851 & -0.526 & 0.000 & 0.000 \\ 0.526 & -0.851 & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.707 & 0.707 \end{bmatrix}$$

Note that the direction of each eigenvector is determined by the direction of just two terms: (t1,t2) or (t3,t4)

matrix

0.000 0.000 1.000 1.000
0.000 0.000 1.000 1.000

tic
coordinates: s1:(t1,t2) and s2:(t3,t4)

Even if t2 does not occur in d2, now if we query with t2 the system will return also d2!!

Co-occurrence space

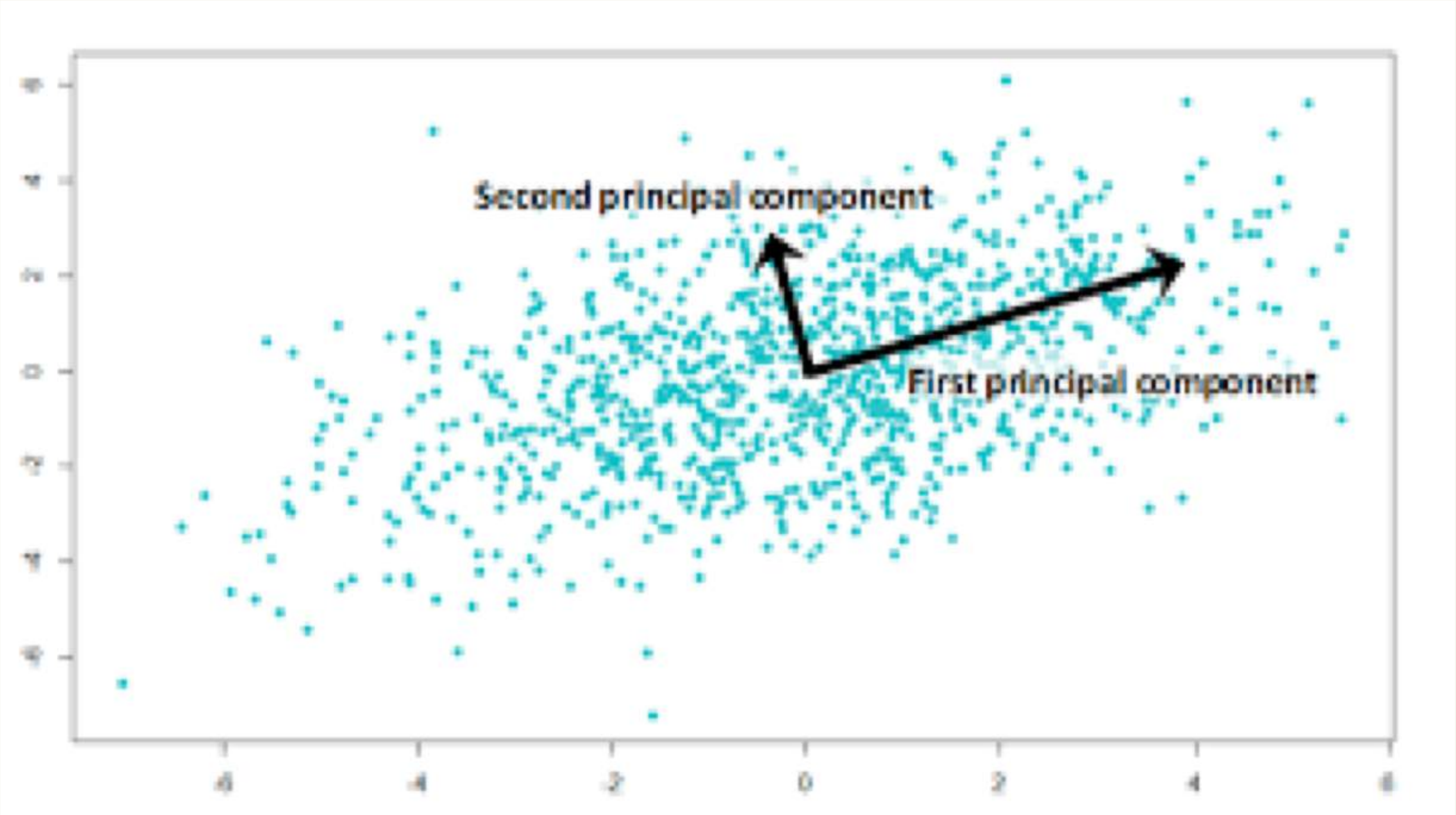
- In principle, the space of document or term co-occurrences is much (much!) higher than the original space of terms!!
- But with SVD we consider only the most relevant ones, trough **rank reduction**

$$A = U \sum V^T \cong U_k \Sigma_k V_k^T = A_k$$

Summary so-far

- We compute the SVD rank-k approximation for the term-document matrix A
- This approximation is based on considering only the *principal eigenvalues* of the term co-occurrence and document similarity matrixes ($L=AA^T$ and $L^T=A^TA$)
- The eigenvectors of the eigenvalues of $L=AA^T$ and $L^T=A^TA$ represent the main "directions"(principal components) identified by term vectors and document vectors, respectively.

Principal components



LSI: what are the steps

- From term-doc matrix A, compute the rank-k approximation $A_{k.}$ with SVD
- **Project docs and queries** in a reduced space of *k<<r* dimensions (the k "survived" eigenvectors) and compute cos-similarity as usual
- These dimensions are **not** the original axes (terms), but those defined by the orthonormal space of the reduced matrix A_k

$$A\vec{q} \cong A_k\vec{q} = \sigma_1 q_1 \vec{v}_1 + \sigma_2 q_2 \vec{v}_2 + \ldots \sigma_k q_k \vec{v}_k$$

Where $\sigma_i q_i$ (i=1,2..k<<*r)* are the new coordinates of q in the orthonormal space of Ak

Projecting terms documents and queries in the LS space

$$(\mathbf{t}_i^T) \rightarrow \overset{\boxed{A}}{\overset{(\mathbf{d}_j)\downarrow}{\begin{bmatrix} x_{1,1} & \dots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \dots & x_{m,n} \end{bmatrix}}} = (\hat{\mathbf{t}}_i^T) \rightarrow \overset{U}{\begin{bmatrix} \begin{bmatrix} \\ \mathbf{u}_1 \\ \\ \end{bmatrix} & \dots & \begin{bmatrix} \\ \mathbf{u}_l \\ \\ \end{bmatrix} \end{bmatrix}} \overset{\Sigma}{\begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_l \end{bmatrix}} \cdot \overset{V^T}{\overset{(\hat{\mathbf{d}}_j)\downarrow}{\begin{bmatrix} [& \mathbf{v}_1 &] \\ & \vdots & \\ [& \mathbf{v}_l &] \end{bmatrix}}}$$

If $A=U\Sigma V^T$ we also have that:

- $V = A^T U \Sigma^{-1}$
- $t = t'^T \Sigma V^T$
- $d = d'^T U \Sigma^{-1}$
- $q = q'^T U \Sigma^{-1}$

After rank k-approximation :

- $A \cong Ak = U_k \Sigma_k V_k^T$
- $d_k \cong d^T U_k \Sigma_k^{-1}$
- $q_k \cong q^T U_k \Sigma_k^{-1}$
- $sim(q, d) = sim(q^T U_k \Sigma_k^{-1}, d^T U_k \Sigma_k^{-1})$

Consider a term-doc matrix MxN (M=11, N=3) and a query q

$$
\begin{array}{l}
\text{Terms} \\
\downarrow \\
\text{a} \\
\text{arrived} \\
\text{damaged} \\
\text{delivery} \\
\text{fire} \\
\text{gold} \\
\text{in} \\
\text{of} \\
\text{shipment} \\
\text{silver} \\
\text{truck}
\end{array}
\qquad
A = \begin{array}{c}
\begin{array}{ccc} d1 & d2 & d3 \\ \downarrow & \downarrow & \downarrow \end{array} \\
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 2 & 0 \\
0 & 1 & 1
\end{bmatrix}
\end{array}
\qquad
\mathbf{q} = \begin{array}{c}
\text{query} \\ \downarrow \\
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1
\end{bmatrix}
\end{array}
$$

1. Compute SVD: $A = U\Sigma V^T$

$$U = \begin{bmatrix} -0.4201 & 0.0748 & -0.0460 \\ -0.2995 & -0.2001 & 0.4078 \\ -0.1206 & 0.2749 & -0.4538 \\ -0.1576 & -0.3046 & -0.2006 \\ -0.1206 & 0.2749 & -0.4538 \\ -0.2626 & 0.3794 & 0.1547 \\ -0.4201 & 0.0748 & -0.0460 \\ -0.4201 & 0.0748 & -0.0460 \\ -0.2626 & 0.3794 & 0.1547 \\ -0.3151 & -0.6093 & -0.4013 \\ -0.2995 & -0.2001 & 0.4078 \end{bmatrix}$$

$$S = \begin{bmatrix} 4.0989 & 0.0000 & 0.0000 \\ 0.0000 & 2.3616 & 0.0000 \\ 0.0000 & 0.0000 & 1.2737 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.4945 & 0.6492 & -0.5780 \\ -0.6458 & -0.7194 & -0.2556 \\ -0.5817 & 0.2469 & 0.7750 \end{bmatrix}$$

$$V^T = \begin{bmatrix} -0.4945 & -0.6458 & -0.5817 \\ 0.6492 & -0.7194 & 0.2469 \\ -0.5780 & -0.2556 & 0.7750 \end{bmatrix}$$

2. Obtain a low rank approximation (k=2) $A_k = U_k \Sigma_k V^T_k$

"latent" 2-dimensional term- similarity space

$$U = \begin{bmatrix} -0.4201 & 0.0748 \\ -0.2995 & -0.2001 \\ -0.1206 & 0.2749 \\ -0.1576 & -0.3046 \\ -0.1206 & 0.2749 \\ -0.2626 & 0.3794 \\ -0.4201 & 0.0748 \\ -0.4201 & 0.0748 \\ -0.2626 & 0.3794 \\ -0.3151 & -0.6093 \\ -0.2995 & -0.2001 \end{bmatrix}$$

$$S = \begin{bmatrix} 4.0989 & 0.0000 \\ 0.0000 & 2.3616 \end{bmatrix}$$

"latent" document- similarity space

$$V = \begin{bmatrix} -0.4945 & 0.6492 \\ -0.6458 & -0.7194 \\ -0.5817 & 0.2469 \end{bmatrix}$$

$$V^T = \begin{bmatrix} -0.4945 & -0.6458 & -0.5817 \\ 0.6492 & -0.7194 & 0.2469 \end{bmatrix}$$

3a. Compute doc/query similarity

- For N documents, A_k has N columns, each representing the coordinates of a document d_i projected in the k LSI dimensions
- A query is considered like a document, and is projected in the LSI space

3c. Compute the query vector

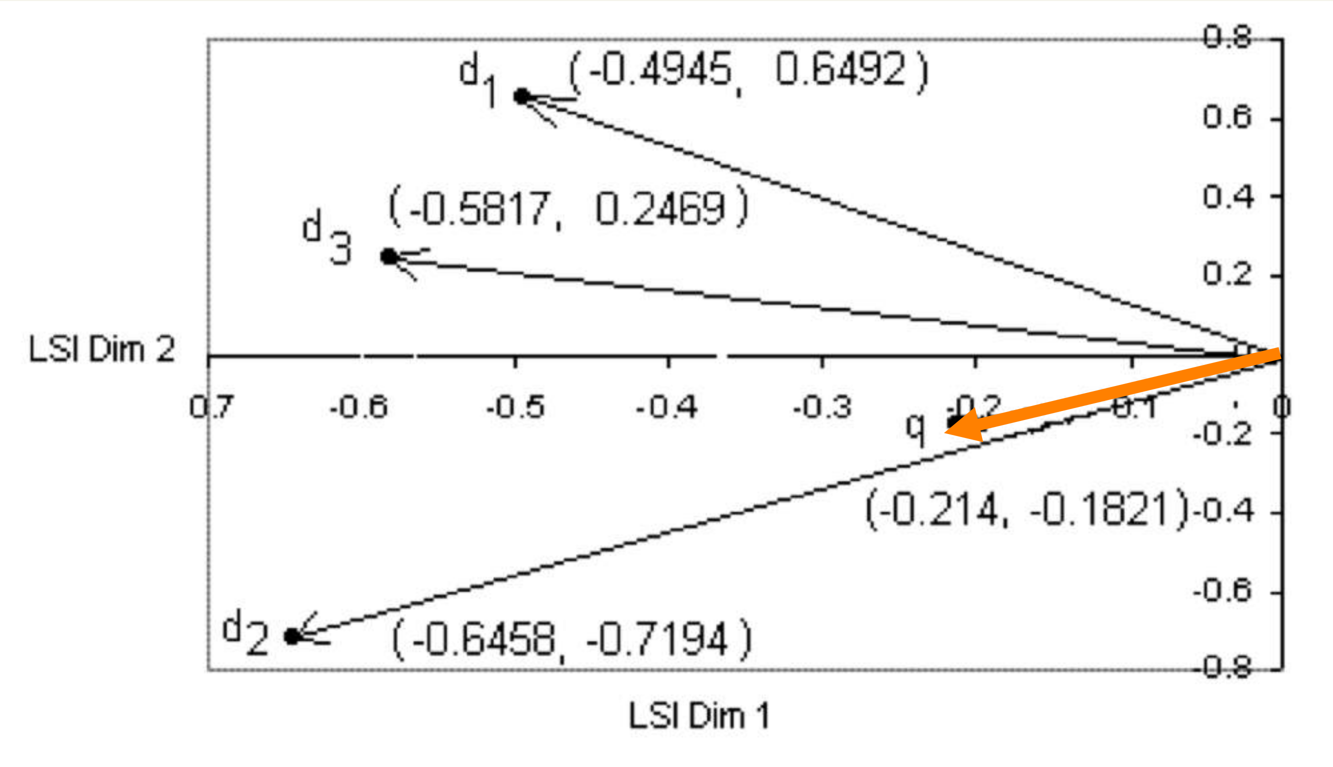
$$q_k = q^T U_k \Sigma_k^{-1}$$

$$\mathbf{q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -0.4201 & 0.0748 \\ -0.2995 & -0.2001 \\ -0.1206 & 0.2749 \\ -0.1576 & -0.3046 \\ -0.1206 & 0.2749 \\ -0.2626 & 0.3794 \\ -0.4201 & 0.0748 \\ -0.4201 & 0.0748 \\ -0.2626 & 0.3794 \\ -0.3151 & -0.6093 \\ -0.2995 & -0.2001 \end{bmatrix} \begin{bmatrix} \frac{1}{4.0989} & 0.0000 \\ 0.0000 & \frac{1}{2.3616} \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} -0.2140 & -0.1821 \end{bmatrix}$$

q is projected in the 2-dimension LSI space!

Documents and queries projected in the LSI space



q/d similarity

$$\text{sim}(q, d) = \frac{q \bullet d}{|q||d|}$$

$$\text{sim}(q, d_1) = \frac{(-0.2140)(-0.4945) + (-0.1821)(0.6492)}{\sqrt{(-0.2140)^2 + (-0.1821)^2}\sqrt{(-0.4945)^2 + (0.6492)^2}} = -0.0541$$

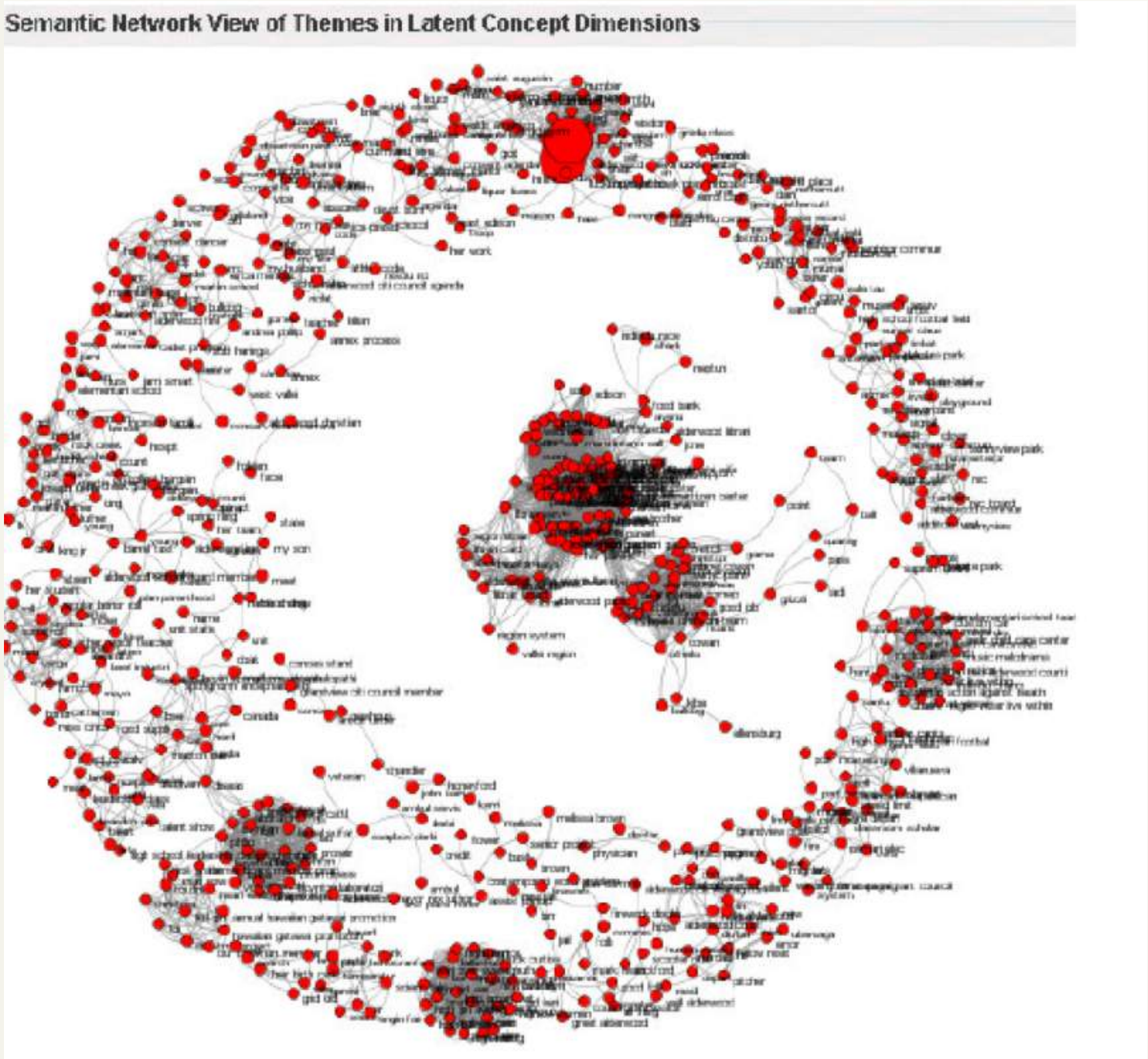
$$\text{sim}(q, d_2) = \frac{(-0.2140)(-0.6458) + (-0.1821)(-0.7194)}{\sqrt{(-0.2140)^2 + (-0.1821)^2}\sqrt{(-0.6458)^2 + (-0.7194)^2}} = 0.9910$$

$$\text{sim}(q, d_3) = \frac{(-0.2140)(-0.5817) + (-0.1821)(0.2469)}{\sqrt{(-0.2140)^2 + (-0.1821)^2}\sqrt{(-0.5817)^2 + (0.2469)^2}} = 0.4478$$

Ranking documents in descending order

$$d_2 > d_3 > d_1$$

An overview of a semantic network of terms based on the top 100 most significant latent semantic dimensions (Zhu&Chen)



Conclusion

- LSI performs a **low-rank approximation** of **document-term matrix** (typical rank **100–300**)
- General idea
  - Map documents (and terms) to a **low-dimensional** representation.
  - Design a mapping such that the low-dimensional space reflects **semantic associations between words** (latent semantic space).
  - Compute document similarity based on the **cos-sim** in this **latent semantic space**

Another LSI Example

Input matrix A:

$$
\begin{array}{c|ccc}
 & d1 & d2 & d3 \\
\hline
t1 & 1.000 & 1.000 & 0.000 \\
t2 & 1.000 & 1.000 & 0.000 \\
t3 & 0.000 & 1.000 & 0.000 \\
t4 & 0.000 & 0.000 & 1.000
\end{array}
$$

Input matrix B:

A^T

$$
\begin{array}{cccc}
1.000 & 1.000 & 0.000 & 0.000 \\
1.000 & 1.000 & 1.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 1.000
\end{array}
$$

AA^T

Matrix product A*B

Term co-occurrences

$$
\begin{array}{c|cccc}
t1 & 2.000 & 2.000 & 1.000 & 0.000 \\
t2 & 2.000 & 2.000 & 1.000 & 0.000 \\
t3 & 1.000 & 1.000 & 1.000 & 0.000 \\
t4 & 0.000 & 0.000 & 0.000 & 1.000
\end{array}
$$

Input matrix:

1.000	1.000	0.000
1.000	1.000	0.000
0.000	1.000	0.000
0.000	0.000	1.000

Singular Value Decomposition:

U:

-0.657	0.000	0.261
-0.657	0.000	0.261
-0.369	0.000	-0.929
0.000	-1.000	0.000

S:

2.136	0.000	0.000
0.000	1.000	0.000
0.000	0.000	0.662

V^T

-0.615	-0.788	0.000
0.000	0.000	-1.000
0.788	-0.615	0.000

$A_k = U_k \Sigma_k V_k^T \approx A$

0.863	1.106	0.000
0.863	1.106	0.000
0.485	0.621	0.000
0.000	0.000	1.000

Now it is like if t3 belongs to d1!

Problems with SVD

- Computational cost **scales quadratically for n x m** matrix: $O(mn^2)$ flops (when n<m)
- Hard to incorporate new words or documents
- Does not consider order of words
- Anything better?
- (note that there are a variety of methods similar to SVD, see "principal component analysis", based on same principles - finding the "main directions" of a set of vectors in a multi-dimensional space)

Is there anything more advanced than co-occurrences to learn correlations?

- Traditional IR uses Term matching, → # of times the doc says “Albuquerque” – not fully appropriate
- We can use a **different approach**: compare all-pairs of query-document **terms**, → # of terms in the doc that relate to Albuquerque
- To detect these similarities (next lessons):
  - Latent Semantic Indexing
  - **Word embeddings (a.k.o. deep method)**

Albuquerque is the most populous city in the U.S. state of New Mexico. The high-altitude city serves as the county seat of Bernalillo County, and it is situated in the central part of the state, straddling the Rio Grande. The city population is 557,169 as of the July 1, 2014, population estimate from the United States Census Bureau, and ranks as the 32nd-largest city in the U.S. The Metropolitan Statistical Area (or MSA) has a population of 902,797 according to the United States Census Bureau's most recently available estimate for July 1, 2013.

Passage *about* Albuquerque

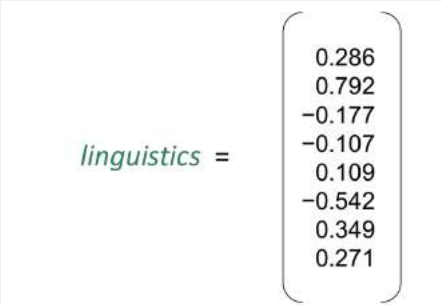
Allen suggested that they could program a BASIC interpreter for the device; after a call from Gates claiming to have a working interpreter, MITS requested a demonstration. Since they didn't actually have one, Allen worked on a simulator for the Altair while Gates developed the interpreter. Although they developed the interpreter on a simulator and not the actual device, the interpreter worked flawlessly when they demonstrated the interpreter to MITS in Albuquerque, New Mexico in March 1975; MITS agreed to distribute it, marketing it as Altair BASIC.

Passage *not about* Albuquerque

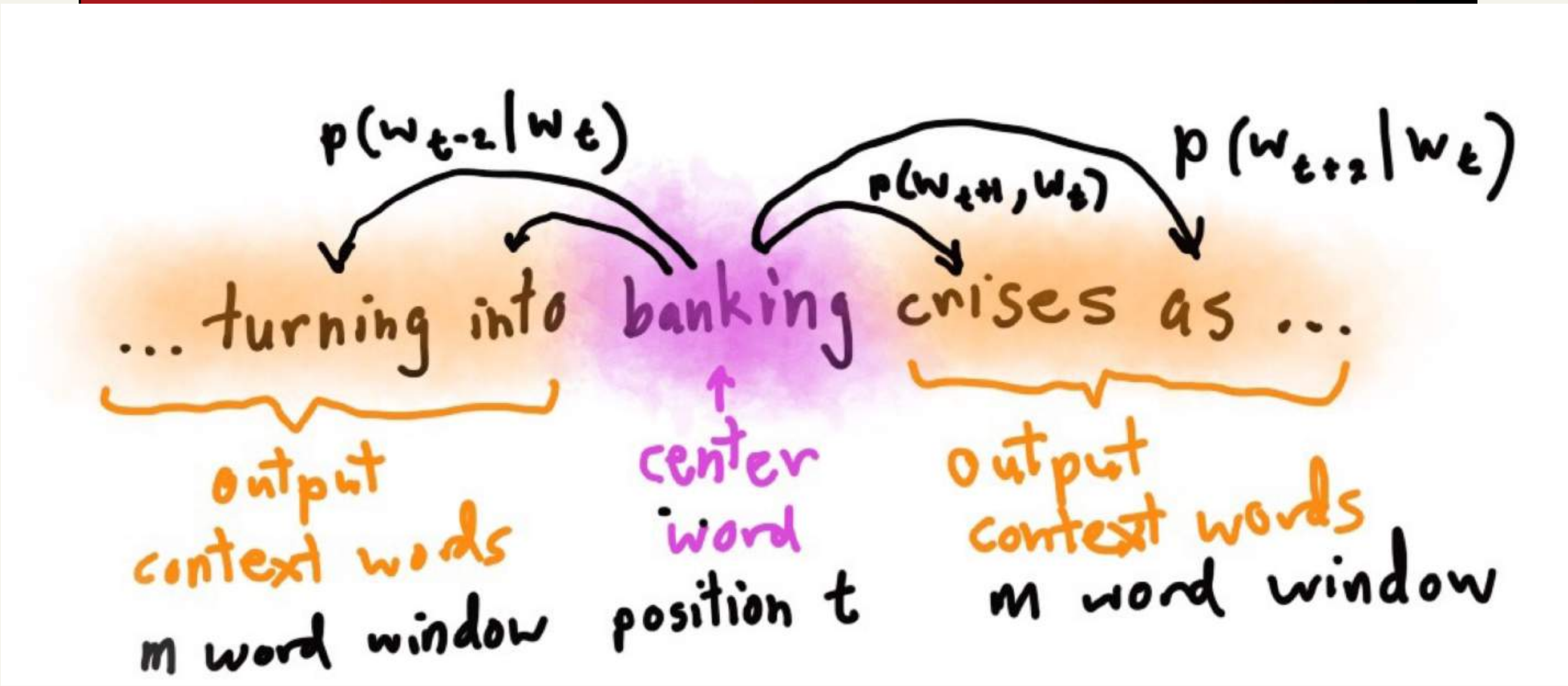
IR with word embeddings

Word Embedding approach: main ideas

- Represent each word with a low-dimensional vector (like for LSI)
- **Word similarity = vector similarity (two words with similar vectors, are similar)**
- Key idea: learn to predict surrounding words in the context of every word, or, learn to predict a word from its surrounding context
- Faster and, wrt SVD, can easily incorporate a new sentence/document or add a new word to the vocabulary



Key idea: semantic similarity among words depends on similarity among **word contexts** in documents



Co-occurrences are considered in a left-right context,
Word ordering DOES matter

Let’s consider the following example…

- We have four (tiny) documents:

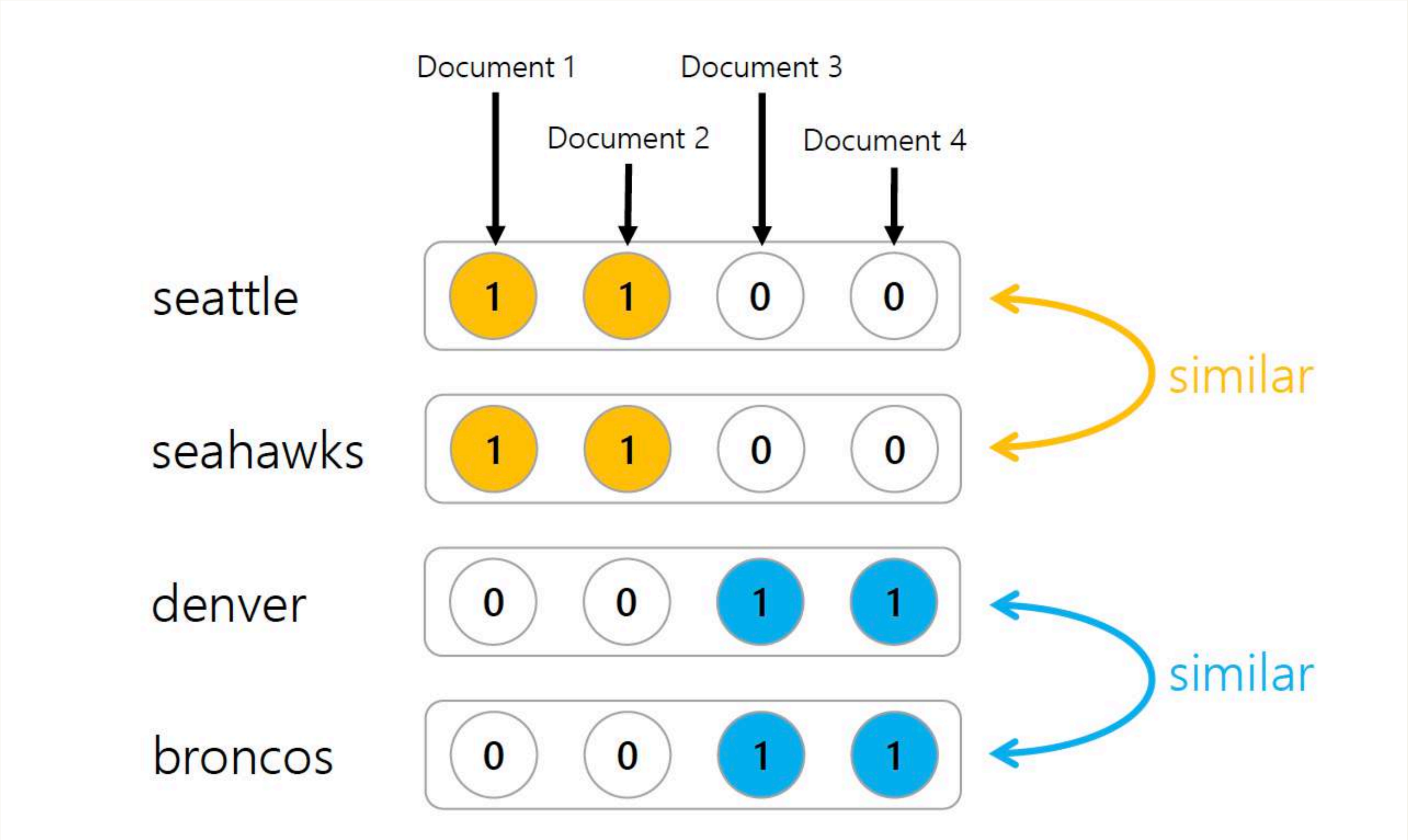
Document 1 : “seattle seahawks jerseys”

Document 2 : “seattle seahawks highlights”

Document 3 : “denver broncos jerseys”

Document 4 : “denver broncos highlights”

Basic difference with previous methods (e.g. LSI with SVD)



SVD would group words based on co-occurrences in documents

If we use context vectors:

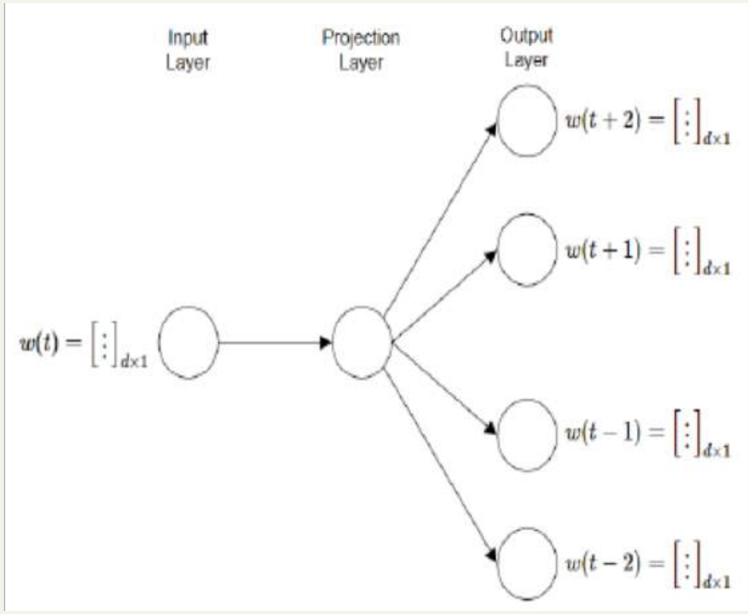
Every position in the vector is a tuple <word, distance from “center” word> an tells us how many times we see that word in the left(right context of a word (e.g. *seahawks* is found 2 times in position +1 to the right of *seattle*) ➔ $p(w_{t\pm i}/w_t)$

Embeddings

- These “context vectors” are very high dimensional (thousands, or even millions) and sparse.
- But there are techniques to learn **lower-dimensional dense vectors** for words using the same intuitions.
- These dense vectors are called **embeddings**.
- Rather than using matrix factorization techniques (such as SVD) we use *deep neural methods*.
- The objective is to represent each word with a dense vector, **such that similar words have similar vectors**
- **We can, as for LSI, consider the dimensions of this dense space as “concepts” or “semantic domains”**

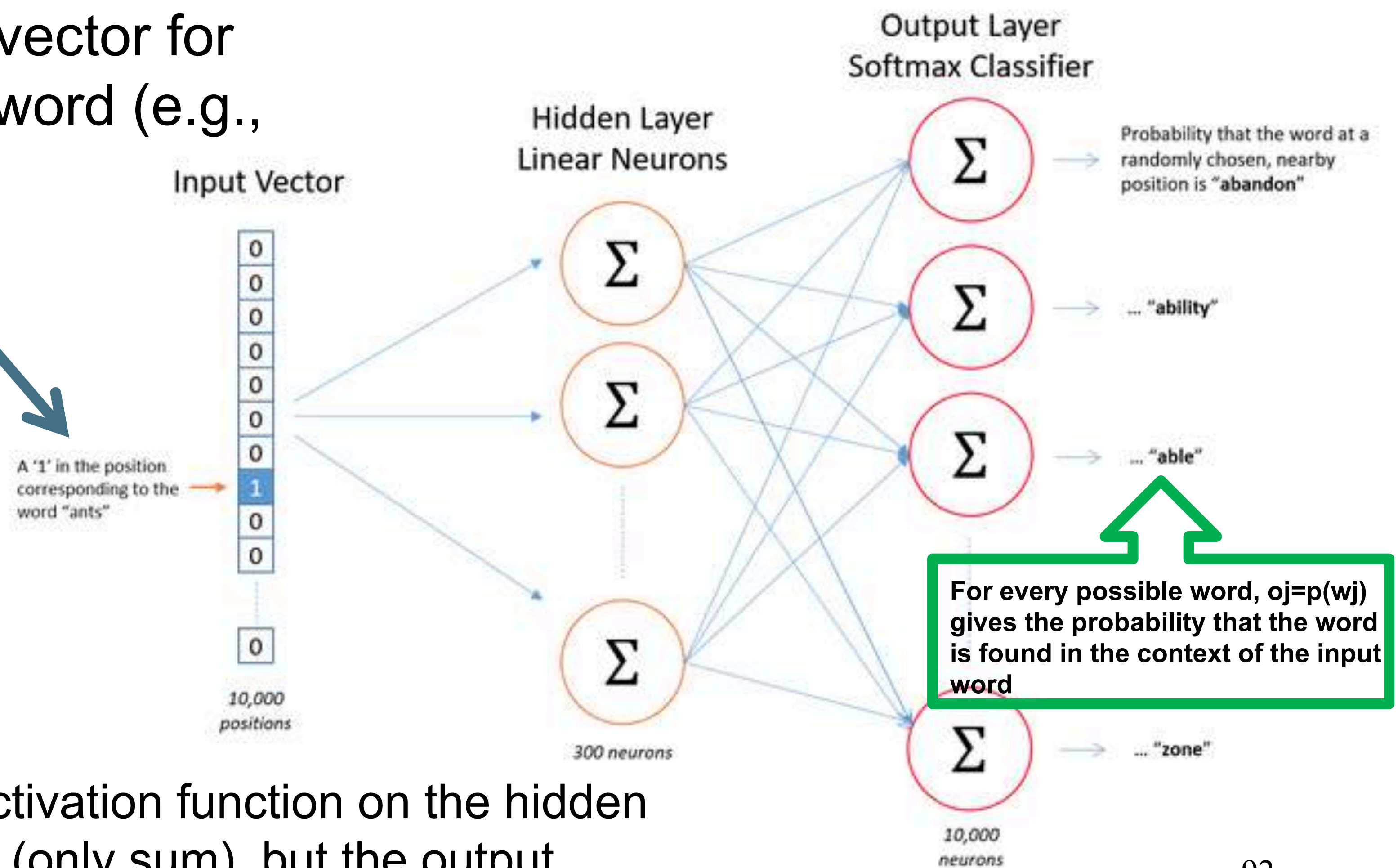
Word Embeddings – Skip Grams Model

- Objective: Given a specific word in the middle of a sentence (the input word w_t), look at the words *nearby* and pick one at random. The neural network should tell us the probability for every word in our vocabulary of being the "nearby word" that we chose.
- "nearby" means that there is a "window size" parameter *m* to the algorithm. A typical window size might be 5, meaning 5 words behind and 5 words ahead (10 in total).
- Our examples hereafter will be with smaller m (1 or 2)



The neural embedding model

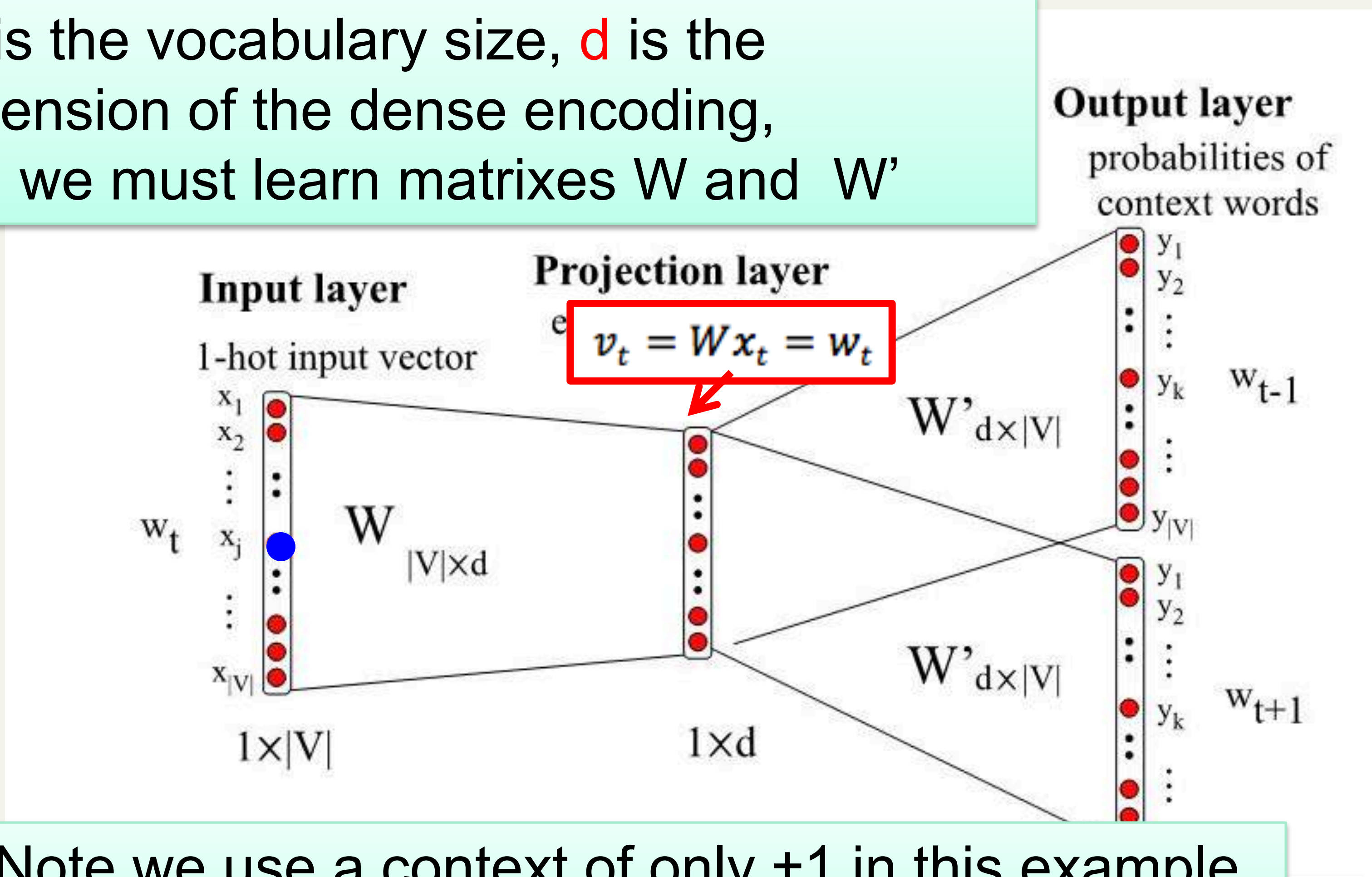
One-hot vector for an input word (e.g., “ants”)



There is no activation function on the hidden layer neurons (only sum), but the output neurons use *softmax*, to output probabilities.

Very same model in terms of matrixes (=the neural weights)

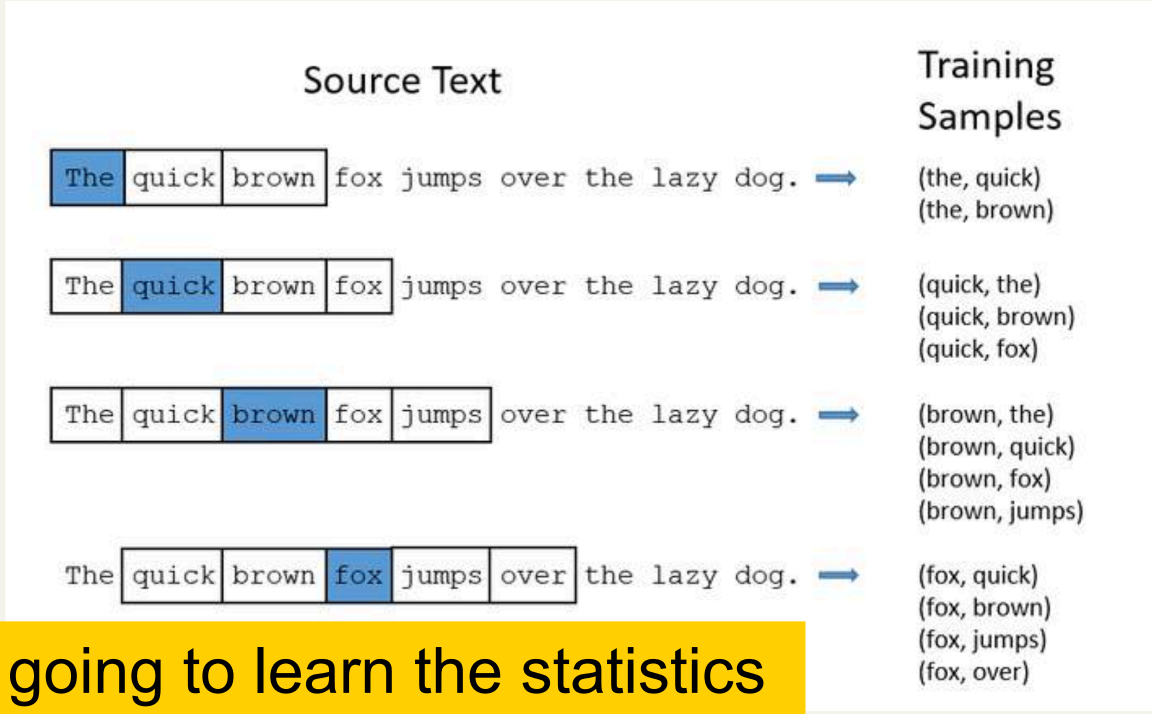
|V| is the vocabulary size, d is the dimension of the dense encoding, and we must learn matrixes W and W'



Note we use a context of only ±1 in this example (one word to the left, one to the right)

Training examples (e.g., for a +-2 window around center)

- The network is trained by feeding it word pairs found in all training documents.



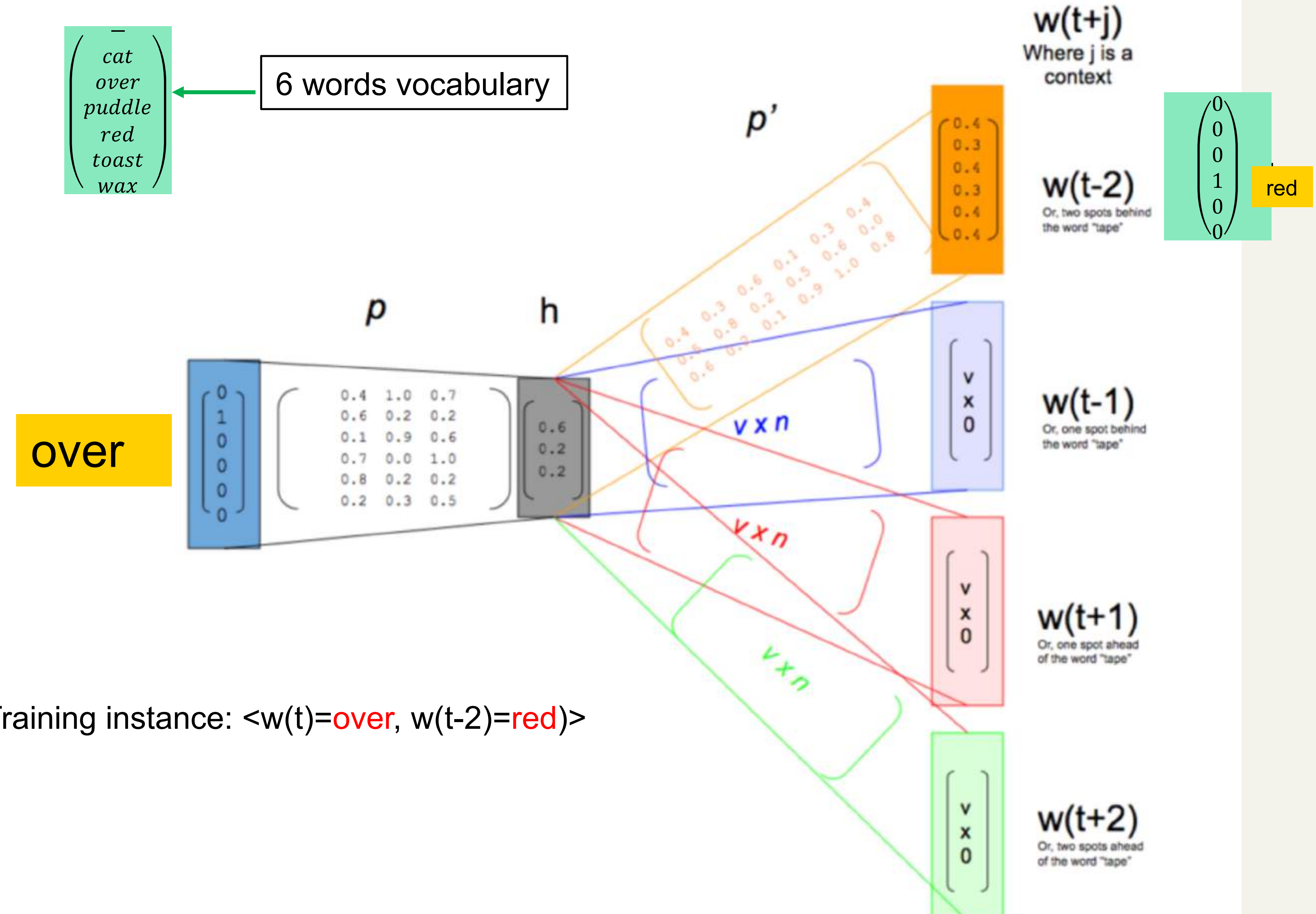
The network is going to learn the statistics from the number of times each "pairing" shows up $P(w_{t\pm i}/w_t)$ (e.g., $P(fox_{+2}/quick)$)

Training steps (1)

- Consider the simple sentence "red cat **over** the puddle". Suppose "over" is the current **center** of the context (our w_t).
- For this example, the input word w_t to the learner is **over**, and the 4 "ground truth" output are "red_{t-2}" "cat_{t-1}" "the_{t+1}" "$puddle_{t+2}$", if the window size is m= ± 2
- We start by generating a "one-hot" vector $\mathbf{x_t}$ for the input, that is, a boolean |V|-dimensional vector with all zeros and a 1 in position t, corresponding to the word "over" in the vocabulary V.
- We obtain the embedded vector by multiplication: $\boldsymbol{v_t = W \times x_t}$
- If *m* is the window size, we generate 2m output vectors using W': $y_{t-m}..\ y_{t-1},\ y_{t+1}..y_{t+m}$ such that $y_j = W'_j \times v_t$
- These vectors are turned into *probability vectors* o_j using softmax ($\Sigma_k\ (o_j^k)=1$). Note that since the matrix *W'* is the same, **all output vectors are equal**! But we update one at the time.

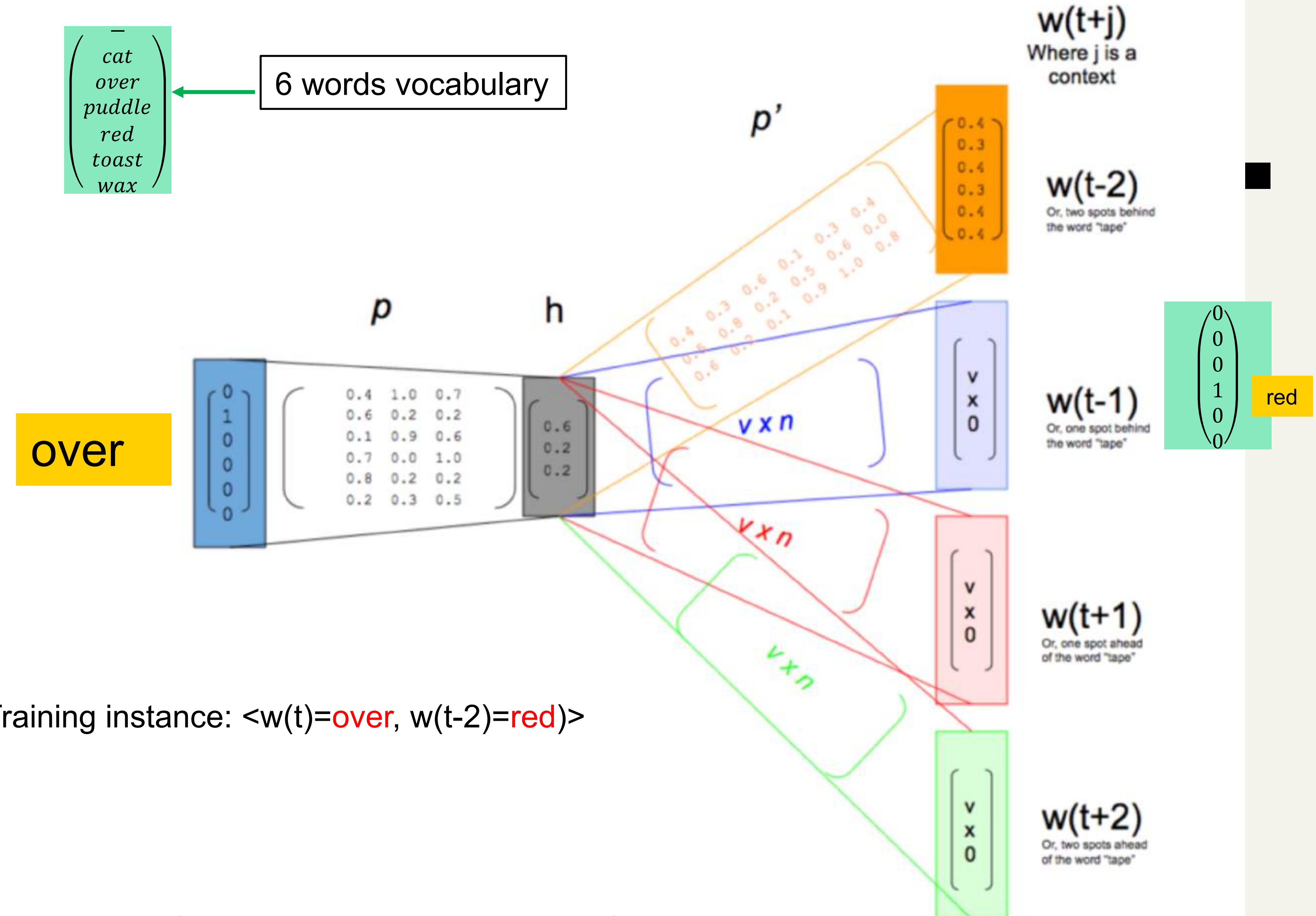
Training step (2)

- Each k^{th} coordinate of any of the 2m softmax output vectors o_j ($1 \times |V|$) represents the probability that the context word at distance $t \pm j$ from our center word w_t is w_k
- We now generate the 2m one-hot vectors x_j corresponding to the current example: for example, in the sentence ”red cat over the puddle” , the one-hot vector x_{t-2}representing the “ground truth” has all zeros and a 1 in the position corresponding to the word “red”
- The one-hot vectors are (one at the time) compared with the generated 2m output vectors o_j, and a loss (error) function is used to update the weights of all matrixes W and W’ (with back-propagation)
- The process is repeated for all sentences and center words until convergence – matrix W and the 2m matrixes W’ no longer change.



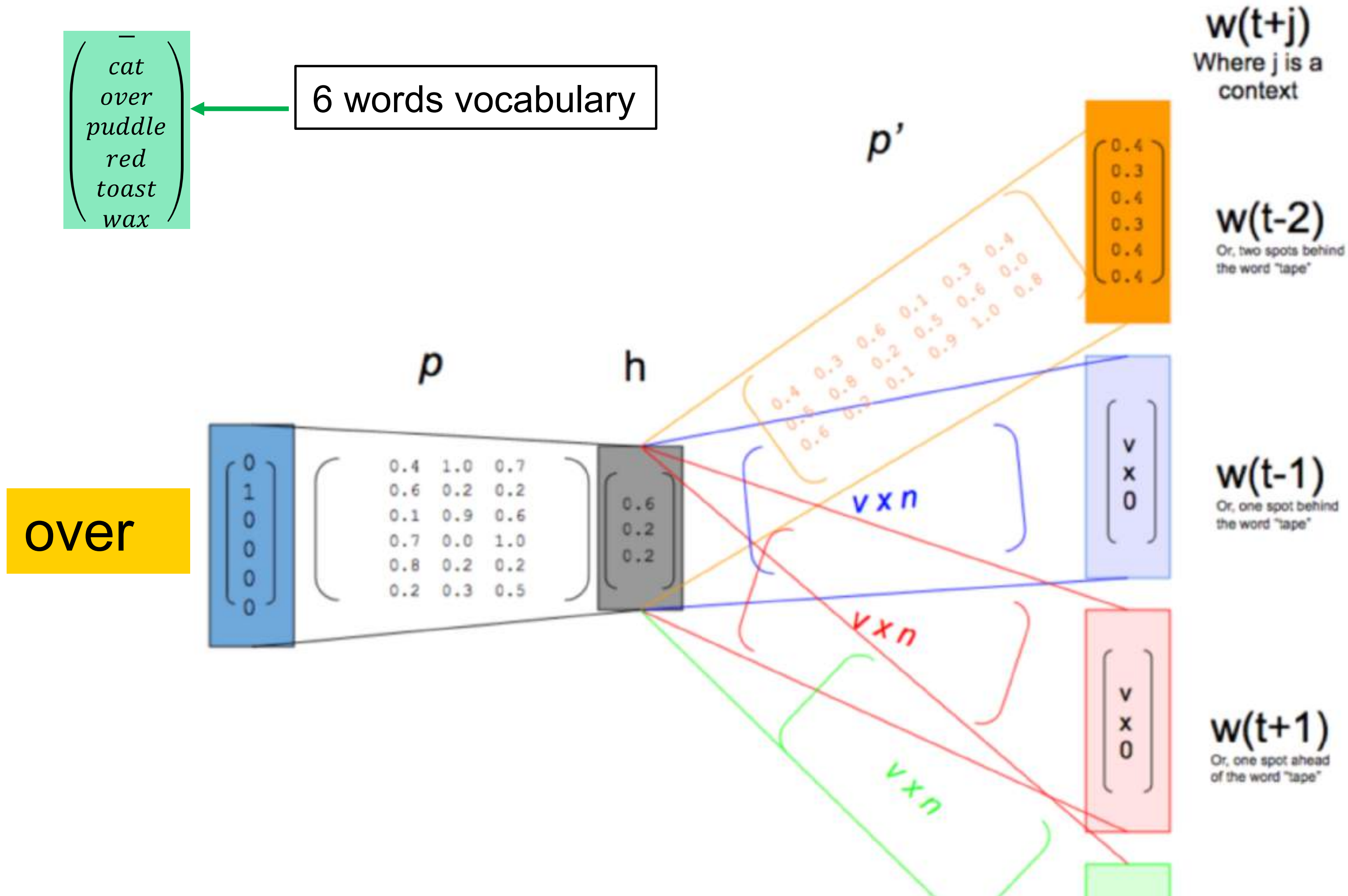
Training instance: <w(t)=over, w(t-2)=red>

We compare ground truth one-hot vector with output vector and backpropagate



Training instance: <w(t)=over, w(t-2)=red)>

We repeat for the second word (note that W and W' have changed after the previous step, although not shown in figure)



We repeat the forward/backword step with the third and fourth training contexts (the$_{+1}$ and puddle $_{+2}$), and then for all other words and contexts

Summary of steps

- We begin by collecting from the corpus the tuples <$w(t),w(t\pm i)$> where w(t)∈ V and *i=1…m*
- The Skip-gram neural net iterates through all words one at a time, with input *w(t)*.
- Each input word *w(t)* is fed forward through the network m*2 times, once for each output context vector (and then again for all retrieved contexts).
- Each time *w(t)* is fed through the network, it is linearly transformed through two weight matrices W and W' to an output layer that contains nodes representing a context location: where m=2, those context locations are from *w(t-2)* to *w(t+2)*.

Summary of steps (2)

- The output nodes, each the size of the vocabulary V, contain scores at each index estimating the likelihood that a word in the vocabulary would appear in that context position.
- For each given training instance, the net will calculate its error between the probability generated for each word in each context location and the observed reality of the words in the context of the training instance.
- For example, the net may calculate that “cat” has a 70% chance of showing up two words before the word “over”, but we can determine from the source corpus that the probability is really 0%. Through the process of backpropagation, the net will modify the weight matrices to change how it projects the input layer through to the output layer in order to minimize its error: for example, to minimize the error between the calculated 70% and the observed 0%.
- Then the next word in the corpus will be sent as an input m*2 times, then the next, and so on.

Additional details

- Suggested reading for embedding algorithms (Skip-grams and CBOW): https://cs224d.stanford.edu/lecture_notes/notes1.pdf
- See also details for the loss function
- As is, summations and weight updating over |V| dimensional matrixes is very time-consuming (the vocabulary is huge, order of millions! And we have millions contexts)
- **Negative sampling** is commonly used: For every training step, instead of looping over the entire vocabulary, we can just sample several negative examples (random word sequences). We "sample" (2m+1) word sequences from a noisy distribution (**P_n(w)**) whose word prior probabilities match the ordering of the frequency of the vocabulary in the corpus.
- With NS, we build a new objective function that tries to maximize the probability of a word and context being in the corpus data if it indeed is, and maximize the probability of a word and context not being in the corpus data if it indeed is not.
- Details on https://arxiv.org/pdf/1310.4546.pdf

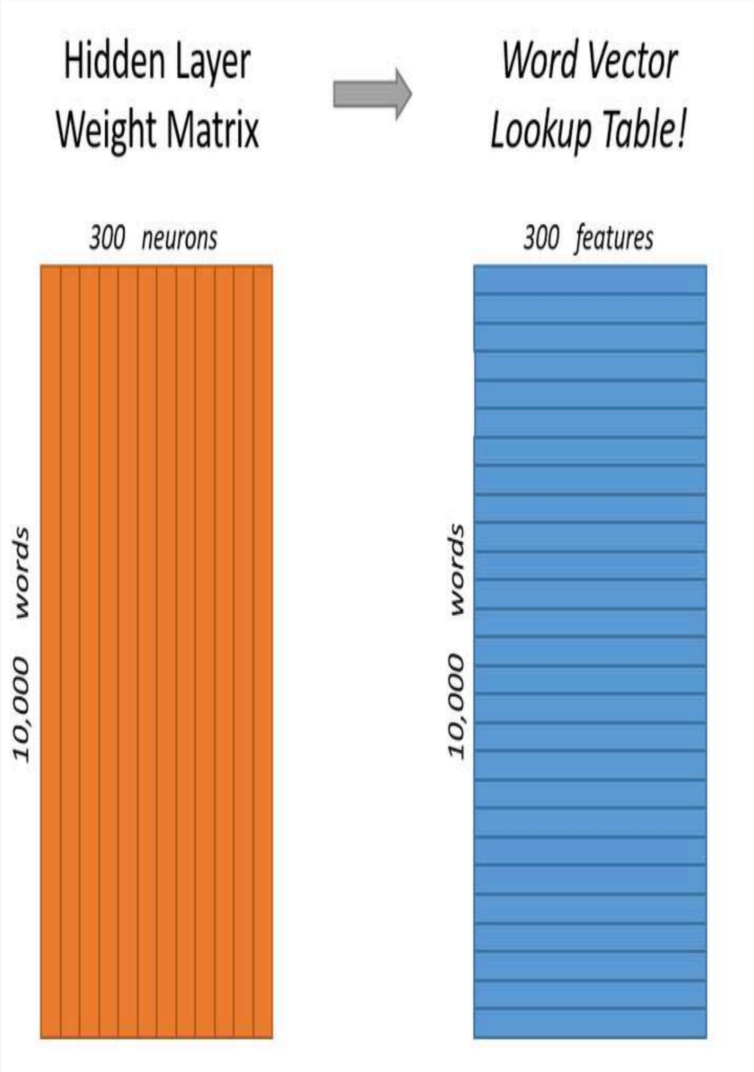
Negative sampling (more on)

- Training a neural network means taking a training example and adjusting all of the neuron weights slightly so that it predicts that training sample more accurately. In other words, each training sample will adjust *all* of the weights in the neural network.
- Negative sampling addresses this by having each training sample only modify a small percentage of the weights, rather than all of them.
- With negative sampling, we randomly select just a small number of “negative” words (let’s say 5) to update the weights for. (here, a “negative” word w_n is one for which we want the network to output a 0 in the correspondent n-th position of output context vectors oj). We will also still update the weights for our “positive” words (e.g., “cat” “puddle” in previous example).
- Negative words are randomly selected

Word embedding hyperparameters

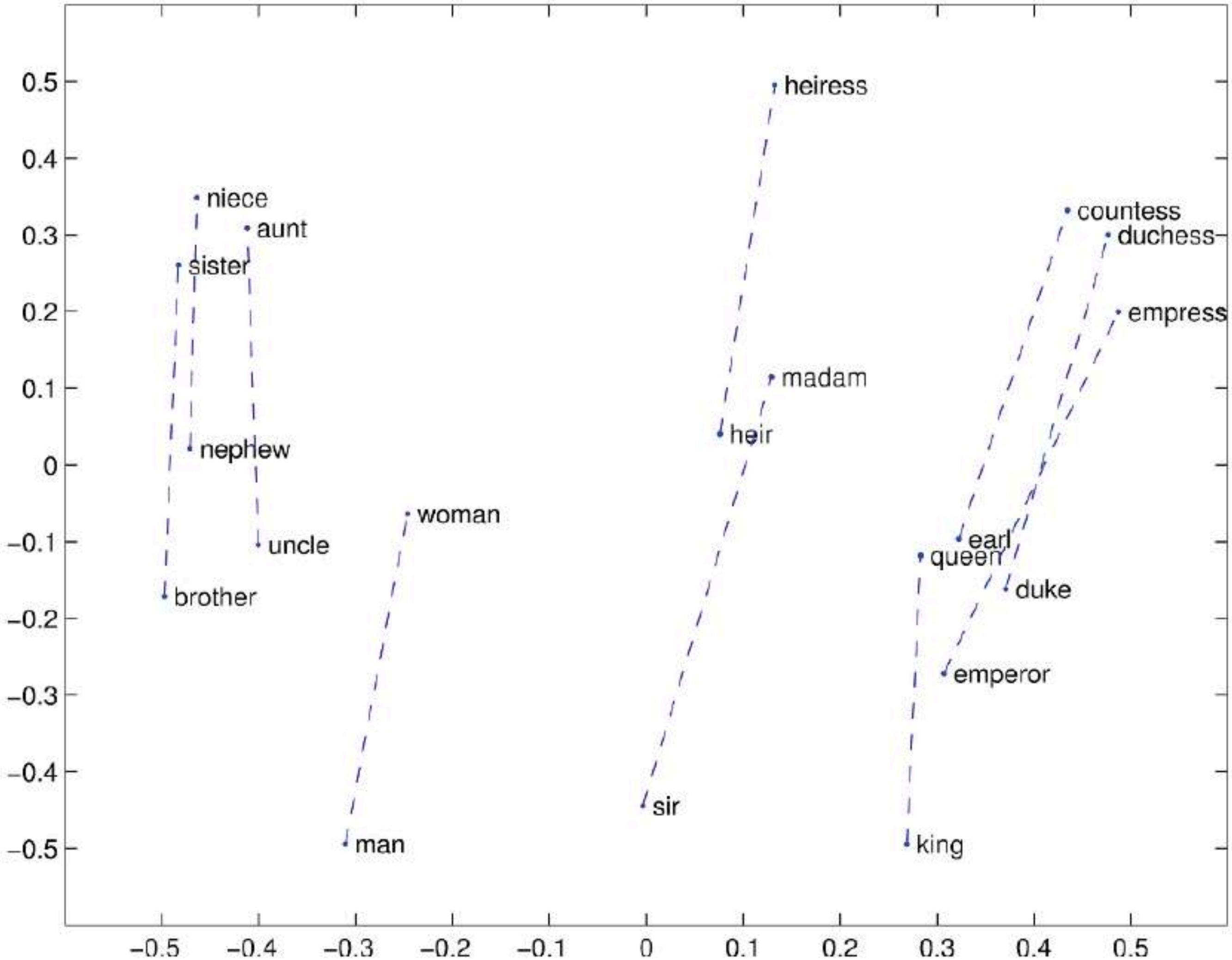
- *V dimension of vocabulary*
- *d dimension of embeddibg vectors*
- *m dimension of context*

Matrixes W and W’

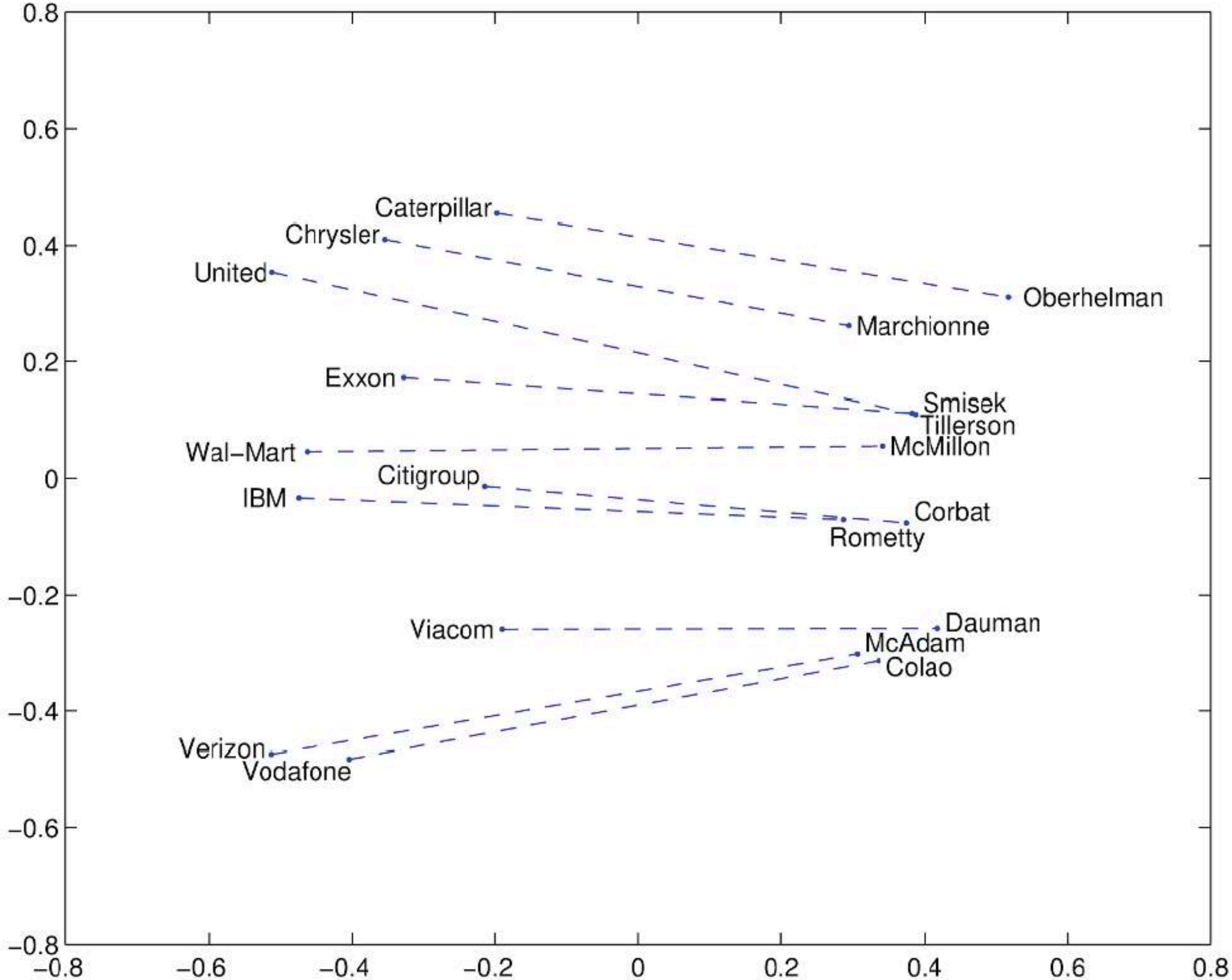


- **Several implementations: word2vect and Glove among the most well known**
- Google word2vect original paper has d=300 and |V|=10,000
- The matrix W is what we are really interested in: **the embedding matrix**.
- It has the property that **words with similar embedding vectors are similar.**

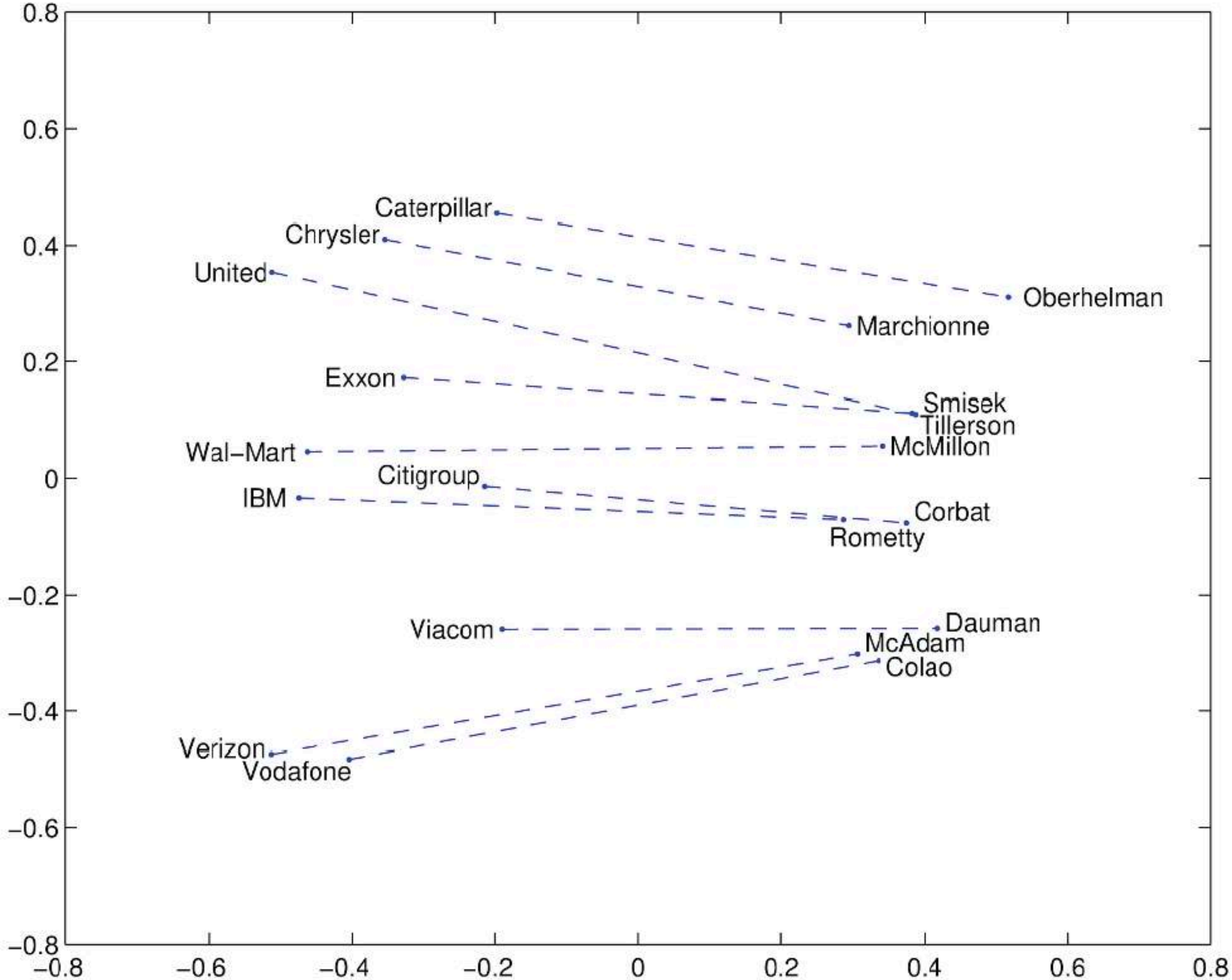
GloVe Visualizations



Glove Visualizations: Company - CEO



Glove Visualizations: Company - CEO



Applications of Word Embeddings to IR

- Word embeddings are the "hot new" technology for document ranking
- Lots of applications wherever knowing word contexts or similarity helps predicting users' interests:
  - **Synonym handling in search**
  - **Query expansion**
  - **Document "aboutness"**
  - Machine translation
  - Sentiment analysis
  - ….

Applications of Word Embeddings to IR: Google RankBrain

- Google's RankBrain – almost nothing is publicly known
  - Bloomberg article by Jack Clark (Oct 26, 2015):
  - http://www.bloomberg.com/news/articles/2015-10-26/google-turning-itslucrative-web-search-over-to-ai-machines
  - A result re-ranking system

Weakness of Word Embedding

- Very vulnerable, and not a robust concept
- Can take a long time to train (despite negative sampling and other “tricks”)
- Non-uniform results
- Hard to understand and visualize
- Emerging technique, yet not sufficiently robust and well understood
- Yet very cool (Google uses it – with other methods)