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Linear Algebra in Geography: Eigenvectors of Networks

Accessibility of towns on a road map is measured by the principal eigenvector of its adjacency matrix.

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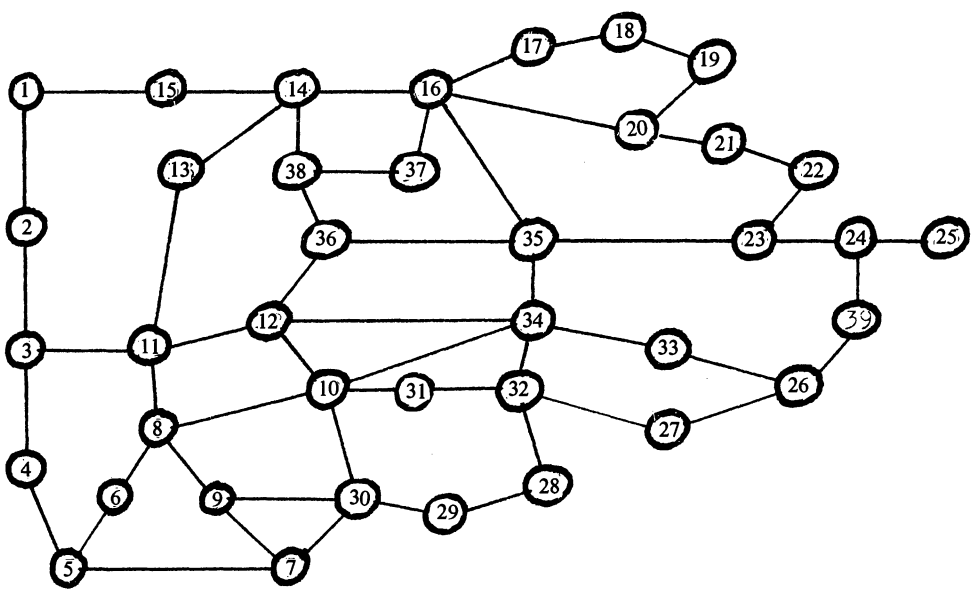
Like economics and psychology before it, modern theoretical geography is a discipline in which the use of mathematics has become increasingly important. In this article I would like to discuss one use of linear algebra in geography. The application is elementary enough to be presented to a first undergraduate linear algebra class, although to my knowledge it has not appeared in linear algebra texts except for a brief mention in [2]. It illustrates well the problem of giving meaningful interpretation to the results of mathematical manipulation of physical data.

The geographical problem starts with a transportation network—a map of geographically significant entities (for instance urban centers) connected by transportation routes (for instance railway lines, highways, or scheduled air routes). Such a network can be conveniently repre-



Trade routes in medieval Russia (adapted from [13]).

Figure 1.

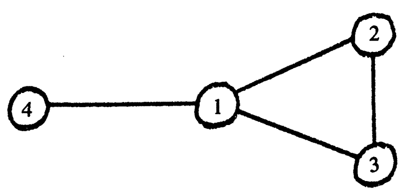
sented by a graph in which the urban centers are vertices and the transportation routes are edges joining pairs of vertices. For example, the graph in FIGURE 1, adapted from [13], represents the major river trade routes in central Russia in the twelfth and thirteenth centuries.

An Index of Accessibility

The problem we address is the development of a suitable index of what geographers have called the accessibility of each vertex in the network. This index should provide a numerical answer to such questions as, "How accessible is this vertex from other vertices in the network? What is its relative geographical importance in the network?" Such an index, once devised, could be used in a number of interesting ways. For instance:

1. Knowledge of which vertices have the highest accessibility could be of interest in itself. For example, the principality of Moscow (number 35 in FIGURE 1) eventually assumed the dominant position in central Russia. Pitts in [13] reviews claims by geographers that this was due to the strategic location of Moscow on medieval trade routes. Does Moscow indeed have the highest accessibility in this network, or must other factors have modified the forces of "situational determinism?"
2. The accessibility of vertices could be statistically correlated to other economic, sociological or political variables to test theoretical hypotheses in geography. Is high accessibility of an urban center in a transportation network associated with high per capita income, a high suicide rate, or a high degree of political awareness?
3. Accessibility indices for the same urban centers in different transportation networks could be compared, as in [4], where the rail network and the interstate highway network are compared for cities in the southeastern United States.
4. Proposed changes in a transportation network could be evaluated in terms of their effect on the accessibility of vertices. Which urban centers would become more, or less, transportationally important?

One solution to the problem of developing a suitable index of accessibility was first proposed by Peter Gould in [6]. Postponing for a moment the justification of his index, let us see how it works. As an example, consider the graph in FIGURE 2.



A simple transportation network.

FIGURE 2.

The **adjacency matrix** of a graph is the square matrix with rows and columns labeled by the vertices, and entries

$$a_{ij} = \begin{cases} 1 \text{ if vertices } i \text{ and } j \text{ are joined by an edge,} \\ 0 \text{ if vertices } i \text{ and } j \text{ are not joined by an edge.} \end{cases}$$

It is traditional to define the diagonal entries a_{ii} to be zero, in which case we will denote the adjacency matrix by A. We will have more occasion to use the modified matrix in which the diagonal entries are defined to be one, and we denote this matrix by $B = A + I$. For example, the adjacency matrices for the graph in FIGURE 2 are

$$A=\begin{bmatrix}0&1&1&1\\1&0&1&0\\1&1&0&0\\1&0&0&0\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}1&1&1&1\\1&1&1&0\\1&1&1&0\\1&0&0&1\end{bmatrix}$$

Gould's definition is based on the eigenvalues of the matrix A. In this example, the characteristic polynomial for A is $\lambda^4-4\lambda^2-2\lambda+1=(\lambda+1)(\lambda^3-\lambda^2-3\lambda+1)$, with approximate eigenvalues $\lambda_1=2.17$, $\lambda_2=.31$, $\lambda_3=-1.00$, $\lambda_4=-1.48$. Now compute the eigenvector $\mathbf{v}_1$ for the largest, or **principal**, eigenvalue λ_1, normalized in any convenient way. In our example we have $\mathbf{v}_1=(.32,.27,.27,.14)$, where the normalization has been done so that the components add to one. The components of this eigenvector are Gould's **index of accessibility**. We note that vertex 1 has the

Node	$\mathbf{v}_1$ (Gould index)	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_{19}$
1. Novgorod	.0042	.0439	.0883	−.1278
2. Vitebsk	.0065	.0020	.0990	−.0233
3. Smolensk	.0186	−.0383	.1630	.1231
4. Kiev	.0104	−.0603	.1551	.1651
5. Chernikov	.0176	−.1352	.2309	−.0894
6. Novgorod Severskij	.0208	−.1156	.1687	.0175
7. Kursk	.0303	−.2133	.2623	−.2009
8. Bryansk	.0547	−.1978	.1974	.0930
9. Karachev	.0386	−.2295	.2528	−.0495
10. Kozelsk	.0837	−.1724	−.0801	.1676
11. Dorogobusch	.0477	−.0519	.1598	−.1167
12. Vyazma	.0722	−.0046	−.1081	−.2917
13. "A"	.0207	.0912	.1534	.0519
14. Tver	.0242	.3146	.2296	.1273
15. Vishnij Totochek	.0082	.1245	.1252	−.0028
16. Ksyatyn	.0321	.4337	.1800	−.0708
17. Uglich	.0105	.1872	.0969	−.3397
18. Yaroslavl'	.0044	.1055	.0660	.0014
19. Rostov	.0047	.1165	.0706	.3400
20. "B"	.0121	.2299	.1133	.0680
21. "C"	.0054	.1118	.0370	−.2554
22. Suzdal	.0068	.0921	−.0194	−.1201
23. Vladimir	.0182	.1533	−.0862	.2309
24. Nizhnij Novgorod	.0076	.0721	−.0968	−.0759
25. Bolgar	.0022	.0251	−.0381	−.3719
26. Isad'-Ryazan	.0142	.0122	−.2117	.1015
27. Pronsk	.0164	−.0079	−.2065	−.0825
28. Dubok	.0178	−.0480	−.1123	−.0769
29. Elets	.0193	−.1033	.0275	.1027
30. Mtsensk	.0493	−.2496	.1822	.0979
31. Tula	.0363	−.0720	−.1547	.2413
32. Dedoslavl'	.0429	−.0349	−.3126	−.1183
33. Pereslavl'	.0267	.0138	−.2095	−.0224
34. Kolomna	.0788	.0275	−.3202	−.1061
35. MOSCOW	.0490	.2772	−.1025	.2432
36. Mozhaysk	.0415	.1837	−.0340	−.0044
37. Dmitrov	.0159	.2396	.1198	−.1132
38. Volok Lamskij	.0234	.2563	.1242	.0476
39. Murom	.0063	.0293	−.1215	.1255
	$\lambda_1=4.48$	$\lambda_2=3.88$	$\lambda_3=3.54$	$\lambda_{19}=1.20$

Eigenvectors for the Russian trade route graph of Figure 1. (Eigenvalues are for the matrix B.)

TABLE 1.

highest index, followed by vertices 2 and 3, followed by vertex 4 with the lowest index, and that these results accord well with intuition. The accessibility indices for vertices in the Russian trade route graph are given in the second column of TABLE 1. Notice that Moscow is not the most highly accessible vertex in the network. In fact it ranks sixth, behind Kozelsk, Kolomna, Vyazma, Bryansk and Mtsensk. The conclusion would be that other sociological and political factors must have been important in Moscow's rise.

It is comforting to have a procedure like the above which seems to reinforce and complement our intuition with numbers carried to several decimal places. However, the first question both geographers and mathematicians must ask is, "What do these numbers mean? Why is it that this manipulation through graphs, matrices, eigenvalues and eigenvectors should produce numbers entitled to the name of an 'accessibility index'?" On this crucial question, Gould and the first users of his index (see **[1]** for example) were unfortunately vague:

> Vectors representing well-connected towns will not only lie in the middle of a large number of dimensions but will tend, in turn, to lie close to the principal axis of our enveloping oblate spheroid. Towns that are moderately well-connected will not lie in the middle of so many dimensions as the well-connected towns, and will tend to form small structural clusters on their own. (**[6]**, page 66)

Although the geometric intuition in this statement tells us something about why the principal eigenvector might have something to do with accessibility, it certainly does not tell us why its components have a claim to giving a precise index. The goal of this article is to use linear algebra to develop three different models to justify Gould's index. We begin with some background from linear algebra.

The Perron-Frobenius Theorem

First, note that the matrix identity $B=A+I$, where I is the $n\times n$ identity matrix, entails that the eigenvalues of B are exactly one larger than the corresponding eigenvalues for A, and that the eigenvectors of the two matrices are exactly the same. Hence we may use the matrix B instead of A to compute Gould's index. Second, the matrix B is symmetric. Linear algebra tells us that a real symmetric matrix can be diagonalized by an orthogonal matrix. Hence all the eigenvalues of B are real, and we can rank them $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The corresponding eigenvectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$ are real and give an orthogonal basis for R^n. Thus the prescription of choosing the largest eigenvalue makes sense, and the components of the principal eigenvector will be real numbers.

The final, crucial piece of information we need is the Perron-Frobenius Theorem for nonnegative square matrices. This theorem is so important for applications of linear algebra in the social sciences that it ought to be at least stated in an undergraduate linear algebra course. Detailed discussions can be found in **[3]** and **[16]**. A matrix $M=(m_{ij})$ is **nonnegative** if $m_{ij}\geqslant 0$ for all i,j. A square nonnegative matrix M is said to be **primitive** if there exists a positive integer k such that all the entries of M^k are strictly positive.

PERRON-FROBENIUS THEOREM. *If M is an $n\times n$ nonnegative primitive matrix, then there is an eigenvalue λ_1 such that*

(i) λ_1 *is real and positive, and is a simple root of the characteristic equation,*
(ii) $\lambda_1 > |\lambda|$ *for any eigenvalue* $\lambda \neq \lambda_1$,
(iii) λ_1 *has a unique (up to constant multiples) eigenvector* $\mathbf{v}_1$, *which may be taken to have all positive entries.*

To apply this theorem to the adjacency matrix of a graph, note that by the definition of matrix multiplication, the ijth entry of A^k counts the number of ways of getting from vertex i to vertex j by paths of length k. The effect of the 1's along the diagonal in $B=A+I$ is to make the ijth entry of B^k count the number of ways of getting from vertex i to vertex j by paths of length k, including possible stopovers at vertices along the way. For a connected graph, the **diameter** of the graph is the smallest integer k such that any vertex may be reached from any other vertex by a path of length less than or equal to k. Hence if our graph is connected and we choose k to be bigger than or equal to its diameter, the entries of B^k will all be positive. In other words, B is primitive, so the Perron-Frobenius Theorem applies. (It is an easy exercise to show that if the underlying graph is **bipartite**, that is, if its vertices can be partitioned into two sets V_1 and V_2 such that no two vertices in V_i are adjacent, $i=1,2$, then the matrix A will *not* be primitive. See **[10]** or **[15]**. It is for this reason that we work with B instead of A.)

Thus if the transportation network is connected, we are guaranteed that the principal eigenvector $\mathbf{v}_1$ is well-defined and has all positive entries. Moreover, consider any vector $\mathbf{x}$ not orthogonal to $\mathbf{v}_1$:

$$\mathbf{x}=\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\cdots+\alpha_n\mathbf{v}_n \quad (\alpha_1\neq 0).$$

Then $B^k\mathbf{x}=\lambda_1^k\alpha_1\mathbf{v}_1+\lambda_2^k\alpha_2\mathbf{v}_2+\cdots+\lambda_n^k\alpha_n\mathbf{v}_n$ and, as $k\to\infty$,

$$\frac{B^k\mathbf{x}}{\lambda_1^k}=\alpha_1\mathbf{v}_1+\left(\frac{\lambda_2}{\lambda_1}\right)^k\alpha_2\mathbf{v}_2+\cdots+\left(\frac{\lambda_n}{\lambda_1}\right)^k\alpha_n\mathbf{v}_n\to\alpha_1\mathbf{v}_1,$$

since λ_1 is the eigenvalue of strictly largest modulus. In other words, *the ratios of the components of $B^k\mathbf{x}$ approach the ratios of the components of $\mathbf{v}_1$ as k increases.* This important fact will provide the basis for our three justifications for using the components of $\mathbf{v}_1$ as an accessibility index.

Justifications of Gould's Index

The first justification of Gould's index relies on the fact mentioned above that the ijth entry of B^k counts the number of paths between vertices i and j of length k (allowing stopovers). A highly accessible vertex should have a large number of paths to other vertices. This idea was used to study accessibility before Gould's article in 1967—see **[4]** and **[13]** for example. The idea was to compute B^k (or often A^k) for some suitably large k (often the diameter of the graph), and then use the row sums of its entries as a measure of accessibility. The accessibility index of vertex i would thus be the sum of the entries in the ith row of B^k, and this is the total number of paths of length k (allowing stopovers) from vertex i to all vertices in the graph. One problem with this method is that the integer k seems arbitrary. At exactly what path length should you stop counting? Here our linear algebra comes to the rescue. Let $\mathbf{e}$ be the n-dimensional column vector all of whose components are 1's. Then the vector of row sums of B^k is just the vector $B^k\mathbf{e}$. Since $\mathbf{e}$ and $\mathbf{v}_1$ both have all positive entries, they cannot be orthogonal, so the ratios of the components of $B^k\mathbf{e}$ approach the ratios of the components of $\mathbf{v}_1$ as k increases. As we count longer and longer paths, this measure of accessibility converges to Gould's index.

A second justification for Gould's index was given by Tinkler in **[17]**. Imagine a rumor starting with a teller at some vertex i in the network at time 0. By time 1, the teller has told the rumor to someone at each vertex which is joined to vertex i by an edge (and of course he remembers it himself). By time 2, each person who knew the rumor at time 1 has told it to someone at each vertex which is adjacent to his vertex. As time progresses, the rumor will spread throughout the network, and we might measure the accessibility of a vertex by the number of people at that vertex who know the rumor. If the spread of a rumor seems too frivolous, think of the spread of a technological innovation, or a trade good.

Once again, our linear algebra is applicable. Let $\mathbf{x}_i$ be the column vector with 1 in the ith position and zeros elsewhere. The distribution of the rumor at time 1 is given by $B\mathbf{x}_i$, and the distribution at time k by $B^k\mathbf{x}_i$. Since $\mathbf{x}_i$ has no negative entries, it cannot be orthogonal to $\mathbf{v}_1$, so the vectors $B^k\mathbf{x}_i$ approach a multiple of $\mathbf{v}_1$ as k increases. No matter where the rumor starts, its equilibrium distribution after a large number of time periods is given by Gould's index.

This model also indicates a geographical meaning of the principal eigenvalue λ_1: it gives the equilibrium growth rate of a rumor spreading according to the model. It is thus a measure of what might be called the "total connectivity" of a network: highly connected networks should be those in which rumors can spread quickly. Perron-Frobenius theory tells us some interesting things about λ_1, for instance that adding an edge to a network must always increase λ_1. For other results on λ_1 and the other eigenvalues of graphs, see **[10]** and **[15]**.

Our third justification of Gould's index is based on an idea discussed by J. W. Moon in **[12]** about how we could measure the relative strengths of players in a round-robin tournament. A first order approximation of the strength of player i might be simply the number of players he beat in the tournament. But beating strong players ought to count more heavily than beating weak players. Hence a second order index of the strength of player i might be the sum of the first order strengths of players he beat. And so it goes: we keep iterating and hope for convergence.

If we apply this idea to a transportation network, it works like this. As a first order measure of the geographical importance of vertex i, we simply use its **degree**, the number of vertices adjacent to vertex i. But being adjacent to important vertices should count more heavily than being adjacent to unimportant vertices. Hence as a second order index of the importance of vertex i, we might use the sum of the first order indices of vertices adjacent to vertex i (and let us count it as adjacent to itself). As we continue the iteration, we recognize what will happen. If $\mathbf{x}$ is the vector of degrees, our kth order index of geographical importance is the vector $B^{k-1}\mathbf{x}$, and these vectors converge to Gould's index as k increases. Gould's index gives the equilibrium relative importance of vertices under this iterative procedure.

Summary

Gould's idea was to measure the accessibility, or geographical importance, of nodes in a transportation network by using the components of the principal eigenvector of the adjacency matrix of the corresponding graph. Originally this idea was justified only on fairly vague heuristic grounds. We have seen that linear algebra, in particular the Perron-Frobenius Theorem, allows us to obtain Gould's index by three separate chains of reasoning. The index gives the relative number of paths joining each vertex to all vertices in the graph, the equilibrium distribution of a rumor spreading in the graph from any vertex, and the equilibrium relative importance of vertices calculated according to an iterative scheme. If we believe that any of these three models captures what we would wish to describe as accessibility, Gould's index is appropriate.

Computing eigenvalues and eigenvectors of large matrices, even when they are sparse, is not an easy task. Hence it is useful to recognize that our analysis yields as a by-product an efficient algorithm, well-known to numerical analysts, for approximating $\mathbf{v}_1$ and λ_1 as closely as desired. Label the columns of an array by the vertices. In the zeroth row, enter a 1 in each column. In the jth column of the $(i+1)$th row, enter the sum of the entries in the ith row corresponding to vertices to which vertex j is adjacent (and count it as adjacent to itself). The rows will converge quite rapidly to the Gould index. The ratio of the total of the $(i+1)$th row to the total of the ith row will converge to the principal eigenvalue of the matrix B. (Recall that this will be one larger than Gould's principal eigenvalue, since he used A instead of B.) The procedure is illustrated in TABLE 2 for the simple graph of FIGURE 2.

Extensions and Generalizations

Gould's index, or its cruder predecessor described above, has been used by geographers to study such things as trade routes in Serbia in the reign of Stefan Dušan **[1]**, the road systems of Uganda in 1921 and of Syria in 1963 **[6]**, the U.S. interstate highway system **[4]**, and the growth of the São Paulo economy **[5]**. See **[17]** and **[20]** for other references to studies of the highway network of northern Ontario, internal migration in Hungary in the 1960's, detribalization in Tanzania, urban accessibility in Indianapolis and Columbus, and the evolution of airline routes

k	1	2	3	4	Total
0	1	1	1	1	4
1	4	3	3	2	12
2	12	10	10	6	38
3	38	32	32	18	120
4	120	102	102	56	380

$$\mathbf{v}_1 \approx (.32, .27, .27., .15) \quad \lambda_1 \approx \frac{380}{120} \approx 3.17$$

Iterative approximation of the principal eigenvalue and eigenvector for the graph in FIGURE 2.

TABLE 2.

in the United States and Australia. In looking at applications, we need not limit ourselves to problems in geography. For instance, indices of the type we have been considering have also been used to study the idea of status in sociology **[11]**, **[8]**. In this context vertices might represent individuals and edges represent friendship or acquaintance. We would be interested in identifying the most "well-connected" individuals, and knowing other well-connected people should count more heavily than knowing poorly connected people. Our third model would suggest the Gould index as an appropriate measure of status.

Several generalizations of Gould's index would be natural. For instance, if enough information were available, we might wish to weight the edges of the transportation graph in some suitable way. Since the weighted adjacency matrix would still be symmetric and primitive nonnegative, the analysis would still work. For example, if we could weight the edges in the Russian trade route network by the volume or worth of trade along various routes, it might turn out that Moscow did have the highest weighted Gould index. For this particular example, though, the historical information necessary for such weighting is not available. The Gould index could also be adapted to directed graphs (one-way trade flows), though then the adjacency matrix would no longer be symmetric, and we would have to require that the digraph be "strongly connected" for the adjacency matrix to be primitive.

In addition to using the principal eigenvector, Gould and other geographers have proposed that the non-principal eigenvectors $\mathbf{v}_2, \mathbf{v}_3, \ldots$ might have geographical meaning. The non-principal eigenvectors must be orthogonal to $\mathbf{v}_1$, which has all positive components. Hence they have some positive and some negative components. Thus in a graph, a non-principal eigenvector partitions the vertices into those with positive components in the eigenvector and those with negative components. This partitioning might pick out significant geographical subsystems. In TABLE 1, the eigenvectors $\mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_{19}$ are given. If you draw the corresponding partitions on FIGURE 1, you will find that $\mathbf{v}_2$ partitions the graph into a northern section and a southern section, $\mathbf{v}_3$ gives an east-west partition, and the partition given by $\mathbf{v}_{19}$ is charmingly complicated. Analyses of this type are given in **[6]** and **[1]**. Going beyond the mystical stage in justifying this kind of analysis seems much more complicated than in the case of the principal eigenvector. Tinkler in **[17]** has proposed an interpretation based on the spread of a rumor (positive numbers) and a canceling anti-rumor (negative numbers) through the network. If the initial distribution of rumor and anti-rumor is exactly given by the components of a non-principal eigenvector corresponding to an eigenvalue $\lambda_j > 1$, then both rumor and anti-rumor will be able to grow at a rate λ_j without either forcing the other out. One problem with this interpretation is lack of stability: if the initial distribution $\mathbf{x}$ differs only slightly from being orthogonal to $\mathbf{v}_1$, we know that eventually its $\mathbf{v}_1$ component will dominate and destroy the coexistence of rumor and anti-rumor. Generically, coexistence is impossible. The nature of the significance, if any, of partitions given by non-principal eigenvectors seems to me as yet unjustified by a reasonable model. Discussions of this question may be found in **[17]**, **[9]** and **[18]**.

I would close by suggesting that the reader interested in getting a feel for how the Gould index works might enjoy calculating Gould indices for some simple families of graphs; for

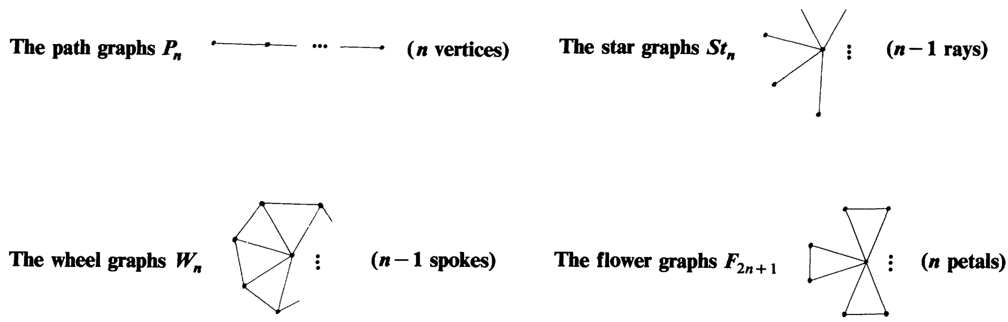


FIGURE 3.

example, those illustrated in FIGURE 3. Some answers appear in [17]. Readers interested in exploring other methods of geographical analysis based on graph theory and linear algebra might consult [7], [14], [19] and [20].

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