Eigenvalues, Eigenvectors and Applications

Intensive Computation

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2020-2021

Lecture 9

INTRODUCTION

Eigenvalue Problems

- Solving linear systems $\boldsymbol{Ax} = \boldsymbol{b}$ is one part of numerical linear algebra, and involves manipulating the rows of a matrix
- The second main part of numerical linear algebra is about find **eigenvalues** and **eigenvectors**
- This is done by *transforming a matrix to leave its eigenvalues unchanged*

Eigenvalue Problems

- The standard algebraic **eigenvalue problem** is:

Given an n x n matrix A, find a scalar λ and a nonzero vector x such that $\mathrm{A}\mathbf{x} = \lambda\mathbf{x}$

where: λ is an ***eigenvalue*** of A

$\boldsymbol{x}$ (non-zero) is the corresponding ***eigenvector***

Example

$$\mathrm{Av} = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda v$$

v=(1,2) is an eigenvector

$\lambda = 5$ an eigenvalue

Eigenvalue Problems

Eigenvalue problems occur in many areas of science and engineering:

- The *natural modes and frequencies of vibration* of a structure are determined by the eigenvectors and eigenvalues of an appropriate matrix
- The *stability of the structure* is determined by the locations of the eigenvalues
- Eigenvalues are useful in *analyzing numerical methods* (the convergence analysis of iterative methods for solving systems of algebraic equations, and the stability analysis of methods for solving systems of differential equations)
- *Graph theory*

Eigenvalue Problems

- An **eigenvector** of a matrix ***determines a direction*** in which the *effect of the matrix is particularly simple*:
  - The ***matrix expands or shrinks*** any vector lying in that direction by a scalar multiple, and
  - the *expansion or contraction factor* is given by the ***corresponding eigenvalue***

- Thus, eigenvalues and eigenvectors provide a means of **understanding the complicated behavior** of a general linear transformation by decomposing it into simpler actions

Eigenvalue Problems

- Although many examples involve only **real matrices**, both the theory and computational procedures are generally applicable to **complex matrices**

- The notation difference for complex matrices is that the conjugate transpose, A^H, is used instead of the transpose, A^T

- The set of all the eigenvalues of a matrix A is called the ***spectrum*** of A and is denoted by $\lambda(A)$

- The maximum modulus of the eigenvalues is called the **spectral radius** of A: $\rho(A) = \max\{|\lambda| : \lambda \text{ in } \lambda(A)\}$

Eigenvalue Problems

- The equation $Ax = \lambda x$ is equivalent to

$$(A - \lambda I)\, x = 0$$

- This homogeneous equation has a **nonzero solution x** if and only if its matrix is **singular**, that is the eigenvalues of A are the values λ such that

$$\mathbf{\det(A - \lambda I) = 0}$$

- $\det(A - \lambda I)$ is a polynomial of degree n in λ:
  - It is the **characteristic polynomial** of A
  - *Its roots are the **eigenvalues*** of A

Eigenvalue Problems

- An nxn matrix A always has n eigenvalues (*Fundamental Theorem of Algebra*)
- Eigenvalues need be neither *distinct* nor *real*
- The product of the eigenvalues is $\det A = \prod_{i=1}^{n} \lambda_i$
- The sum of the eigenvalues is $\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$ called ***trace***

POWER METHOD

Computing Eigenvalues and Eigenvectors

- Many numerical methods for **computing eigenvalues and eigenvectors** are based on **reducing the original matrix** to a simpler form, whose eigenvalues and eigenvectors are then **easily determined**

- **Finding the eigenvalues and eigenvectors** is equivalent to **transforming** the underlying system of equations into a special set of coordinate axes in which the **matrix is diagonal**

- The **eigenvalues** are the entries of the **diagonal** matrix

- The eigenvectors are the new set of coordinate axes

Similarity Transformations

- We need to identify:
  - what types of transformations preserve eigenvalues
  - for what types of matrices the eigenvalues are easily determined

- A matrix B is ***similar*** to a matrix A if there is a nonsingular matrix T such that

$$B = T^{-1}AT$$

- Then $By = \lambda y \rightarrow T^{-1}ATy = \lambda y \rightarrow A(Ty) = \lambda(Ty)$

  so that A **and** B **have the same eigenvalues**, and if y is an eigenvector of B, then $x = Ty$ is an eigenvector of A

Similarity Transformations

- Similarity transformations:
  - Preserve eigenvalues
  - Do not preserve eigenvectors
  - But the eigenvectors are still easily recovered

- Note that the converse is not true
  - two matrices that are **similar** must have the **same eigenvalues**
  - ***but*** two matrices that have the same eigenvalues are ***not necessarily similar***

Similarity Transformations

- The **eigenvalues of a diagonal matrix are its diagonal entries**, and the eigenvectors are the corresponding columns of the identity matrix I
- Note that:
  - Diagonal form simplifies eigenvalue problems for general matrices by similarity transformations
  - But **some matrices cannot be transformed into diagonal form** by a similarity transformation
- Fortunately:
  - **every matrix** can be transformed into **triangular** by a similarity transformation
  - The **eigenvalues of a triangular matrix are also the diagonal entries**

Computing Eigenvalues and Eigenvectors

- There are *several methods* designed to compute all of the *eigenvalues* of a matrix and require a great deal of work

- In practice, one may need ***only one or a few eigenvalues*** and corresponding eigenvectors

- The simplest method for computing a single eigenvalue and eigenvector of a matrix is the **power method**, which takes ***successively*** *higher powers of the matrix* times *an initial starting vector*

Power Method

- Assume that the matrix has a **unique eigenvalue λ_1 of maximum modulus**, with corresponding eigenvector u_1
- Let us consider the following ***iteration scheme***, starting from a given nonzero vector x_0

$$x_k = Ax_{k-1}$$

- The iteration scheme **converges** to a multiple of u_1, the eigenvector corresponding to the dominant eigenvalue λ_1

Power Method

- In fact, if we express the starting vector x_0 as a linear combination, $x_0 = \sum_{i=1}^{n} \alpha_i u_i$ where u_i are eigenvectors of A

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$$= \sum_{i=1}^{n} \alpha_i A^k u_i = \sum_{i=1}^{n} \lambda_i^k \alpha_i u_i = \lambda_1^k (\alpha_1 u_1 + \sum_{i=2}^{n} (\lambda_i/\lambda_1)^k \alpha_i u_i)$$

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- Since $|\lambda_i / \lambda_1| < 1$ for $i > 1$, successively higher powers go to zero, leaving only the component corresponding to u_1

Power Method

- Then, having $x_k = \sum_{i=1}^{n} \lambda_i^k \alpha_i u_i = \lambda_1^k (\alpha_1 u_1 + \sum_{i=2}^{n} (\lambda_i/\lambda_1)^k \alpha_i u_i)$

- We can obtain

$$\frac{\|x_{k+1}\|}{\|x_k\|} = \frac{\|Ax_k\|}{\|x_k\|} \to \frac{\|\alpha_1 \lambda_1^{k+1} u_1\|}{\|\alpha_1 \lambda_1^k u_1\|} = \frac{\alpha_1 \lambda_1^{k+1} \|u_1\|}{\alpha_1 \lambda_1^k \|u_1\|} = |\lambda_1| \text{ as } k \to \infty$$

Power Method

- To avoid eventual overflow (or underflow if the dominant eigenvalue is less than 1 in magnitude), it is better to **normalize** the approximate eigenvector at each iteration
- We can require its largest component to have modulus 1

- This step gives the **iteration scheme**

$$y_k = Ax_{k-1}$$
$$x_k = y_k / \|y_k\|_\infty = Ax_{k-1} / \|Ax_{k-1}\|_\infty$$

- With this normalization $\|y_k\|_\infty \to |\lambda_1|$ and $x_k \to u_1 / \|u_1\|_\infty$

Deflation Methods

- Suppose that an eigenvalue λ_1 and corresponding eigenvector x_1 for a matrix A have been computed
- We can **compute additional eigenvalues** $\lambda_2, \ldots, \lambda_n$ of *A*, by a process called **deflation**, which removes the known eigenvalue
- We construct a new matrix *B* with eigenvalues $\lambda_2, \ldots, \lambda_n$, that is we *deflate* the matrix *A*, removing λ_1
- Then λ_2 of matrix *A* can be obtained matrix *B* by the power method

Deflation Methods

- Let H be any nonsingular matrix such that $Hx_1 = \alpha e_1$, that is H is such that α a scalar multiple of the first column e_1 of the identity matrix I

- Then the **similarity transformation** determined by H transforms A into the form

$$HAH^{-1} = \begin{bmatrix} \lambda_1 & b^T \\ 0 & B \end{bmatrix}$$

where B is a matrix of order n - 1 having eigenvalues $\lambda_2, \ldots, \lambda_n$

- For example, a good choice for H can be an appropriate *Householder transformation* *(linear transformation that describes a reflection about a plane or hyperplane containing the origin)*

Deflation Methods

- We use B to compute next eigenvalue λ_2 and eigenvector y_2
- Once computed y_2, eigenvector of B, we want to compute the second eigenvector x_2 of matrix A

Deflation Methods

- We use B to compute next eigenvalue λ_2 and eigenvector y_2
- Once computed y_2, eigenvector of B, we want to compute the second eigenvector x_2 of matrix A
- We need to add an element to vector y_2 (that consist of n-1 elements), that is $\boldsymbol{x_2 = (s_2\ y'_2)}$
- s_2 can be the element α such that

$$x_2 = H^{-1}\begin{bmatrix}\alpha \\ y_2\end{bmatrix} \quad \text{where} \quad \alpha = \frac{b^T y_2}{\lambda_2 - \lambda_1}$$

- Hence, x_2 is an eigenvector corresponding to λ_2 for the original matrix A, provided that $\lambda_2 \neq \lambda_1$
- Process can be repeated to find additional eigenvalues and eigenvectors

Deflation Methods

Hotelling deflation

- We have: A given, λ_1 and u_1 known (e.g., by power method)
- Consider $B = A - \lambda_1 u_1 u_1^T$

- We can verify that B has the same eigenvectors as $\boldsymbol{A}$, and the same eigenvalues as $\boldsymbol{A}$ except that the largest one has been replaced by 0
- In fact

$$(A - \lambda_1 u_1 u_1^T) u_j = A u_j - \lambda_1 u_1 u_1^T u_j = \lambda_j u_j - \lambda_1 u_1 u_1^T u_j$$

$$\text{If } j = 1 \quad \text{then} \quad (A - \lambda_1 u_1 u_1^T) u_1 = \lambda_1 u_1 - \lambda_1 u_1 (u_1^T u_1) = 0 u_1$$

$$\text{If } j \neq 1 \quad \text{then} \quad (A - \lambda_1 u_1 u_1^T) u_j = \lambda_j u_j - \lambda_1 u_1 (0) = \lambda_j u_j$$

- Then we compute the value of λ_2 by the power method

Deflation Methods

An **alternative approach** is the following:

- A given, λ_1 and x_1 known (e.g., by power method)
- Denote by a^T the first row of A (or the p-th row of A), i.e., $a^T=(a_{11}, a_{12}, \ldots, a_{1n})$
- Consider $B = A - x_1^* a^T$
- where x_1* is the vector x_1 normalized by dividing by its first element (and product x_1* a^T is an nxn matrix)
- We can verify that $B\, x_1=0$ that is 0 is eigenvalue of B (instead of λ_1) and $\lambda_2, \ldots, \lambda_n$ are still eigenvalues (of A)
- Then we compute the value of λ_2 by the power method

Deflation Methods

A **variant** of the previous **alternative approach** is:

- A given, λ_1 and x_1 known (e.g., by power method)
- Denote by v_1 any vector such that $v_1^T x_1 = \lambda_1$
- Then the matrix $A - x_1 {v_1}^T$ has eigenvalues of $0, \lambda_2, \ldots, \lambda_n$
- Then we compute the value of λ_2 by the power method
- There are several possible choices for v_1

Smallest eigenvalue

- For some applications, the **smallest eigenvalue** of a matrix is required rather than the largest
- We can use the property that the *eigenvalues of A^{-1} are the reciprocals of those of A*
- Hence the smallest eigenvalue of A is the reciprocal of the largest eigenvalue of A^{-1}
- In fact

$$\min_{i=1,\dots,n} |\lambda_i(A)| = \min_{i=1,\dots,n} \left| \frac{1}{\lambda_i(A^{-1})} \right| = \frac{1}{\max_{i=1,\dots,n} |\lambda_i(A^{-1})|}$$

- We therefore use the inverse iteration scheme

$$Ay_k = x_{k-1} \Rightarrow y_k = A^{-1}x_{k-1}$$

Convergence

- The **convergence rate** of the power method depends on the ratio $|\lambda_2|/|\lambda_1|$, where λ_2 is the eigenvalue having second-largest modulus
- **The smaller $|\lambda_2|/|\lambda_1|$, the faster the convergence**
- Hence the power method will converge
  - Quickly if $|\lambda_2|/|\lambda_1|$ is small
  - Slowly if $|\lambda_2|/|\lambda_1|$ is close to 1

EXAMPLES

Gould Index of Accessibility

- https://www.math.washington.edu/~morrow/336_11/papers/leo.pdf
- http://matrixapps.blogspot.it/2010/07/gould-index-matrix-application-to.html

- The method proposed by Peter Gould (1967), also known as eigenvector centrality, is one method of computing the *centrality*, or *approximate importance*, of each node in a graph

- The assumption is that each node's centrality is the sum of the centrality values of the nodes that it is connected to

Gould Index of Accessibility

- Let us look at the method

- We begin with the adjacency matrix A of the graph
- It is usual to define the entries a_{ii}, the diagonal, as 0
- We replace the diagonal zeros with ones

- The index that Gould uses the (*normalized*) **eigenvector from the principle eigenvalue** of the **modified adjacency matrix $B=A+I$**

- The i-th entry corresponds to the i-th vertex and this is its **accessibility rank**

Gould Index of Accessibility

- Consider a graph that represents a set of towns (the vertices) and the travel routes between those towns (the edges)
- Historical geographers were interested in which town would become the trade center for this region

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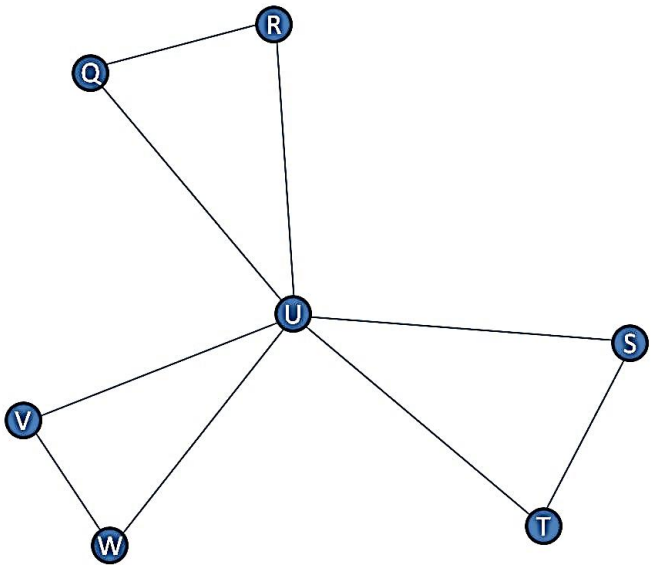
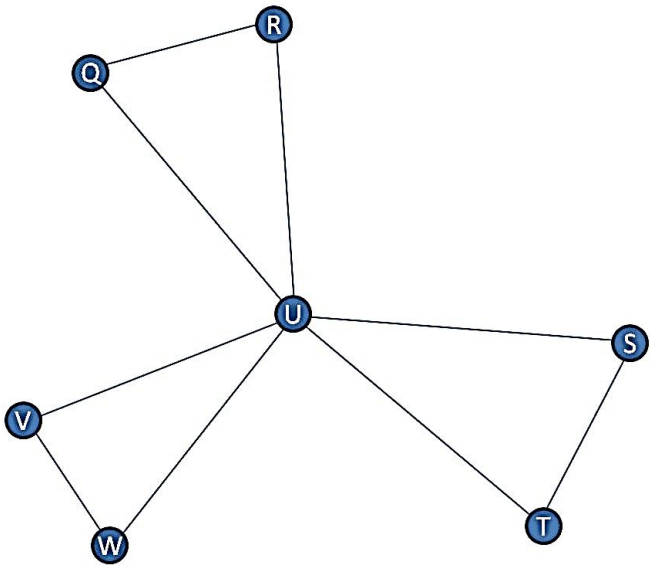


	Q	R	S	T	U	V	W	
	1	1	0	0	1	0	0	Q
	1	1	0	0	1	0	0	R
	0	0	1	1	1	0	0	S
	0	0	1	1	1	0	0	T
	1	1	1	1	1	1	1	U
	0	0	0	0	1	1	1	V
	0	0	0	0	1	1	1	W

Gould Index of Accessibility

- Find the **largest eigenvalue** of the matrix
- The eigenvalues are: (2, 0, 4, -1, 0, 2, 0),
- The third eigenvalue has the largest absolute value - **4**



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	1	1	0	0	1	0	0	R
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	0	0	1	1	1	0	0	T
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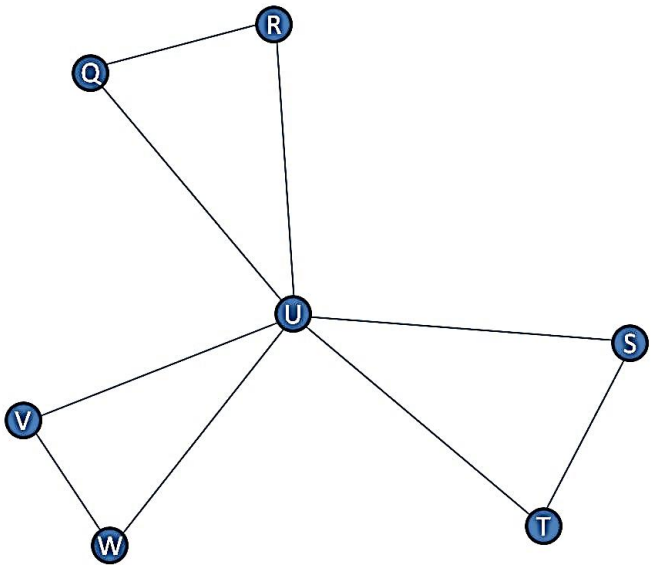
Gould Index of Accessibility

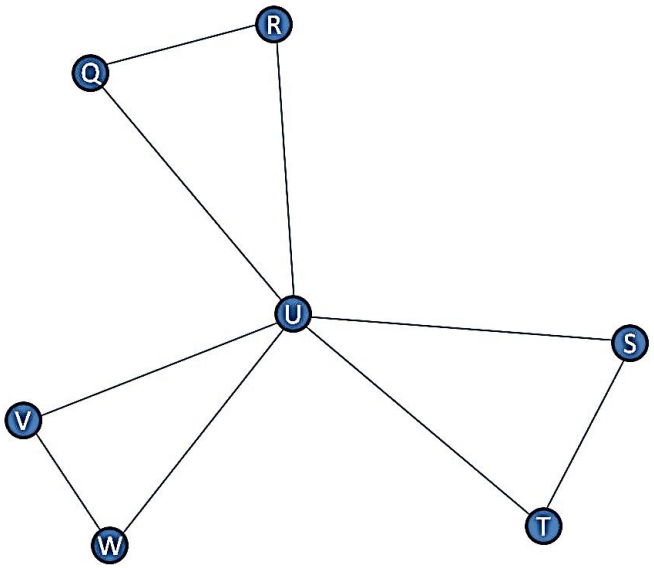
- Find the **largest eigenvalue** of the matrix
- The eigenvalues are: (2, 0, 4, -1, 0, 2, 0),
- The third eigenvalues has the largest absolute value, **4**
- Find the **eigenvector associated** with the eigenvalue of 4: (0.3162, 0.3162, 0.3162, 0.3162, 0.6325, 0.3162, 0.3162)



	Q	R	S	T	U	V	W
Q	1	1	0	0	1	0	0
R	1	1	0	0	1	0	0
S	0	0	1	1	1	0	0
T	0	0	1	1	1	0	0
U	1	1	1	1	1	1	1
V	0	0	0	0	1	1	1
W	0	0	0	0	1	1	1

Gould Index of Accessibility

- You can normalize this vector by dividing by the sum of the entries, 2.5297. You get:
  (Q, R, S, T, U, V, W) = (0.125, 0.125, 0.125, 0.125, 0.25, 1.25, 1.25)



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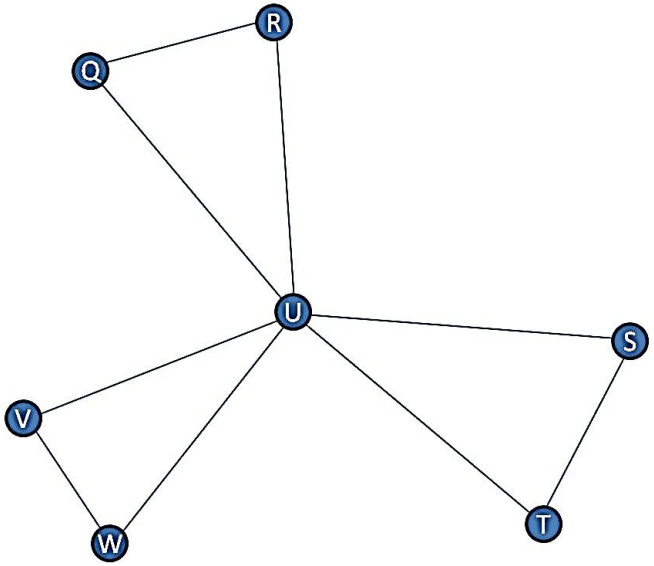
- These are the **Gould indices of each of the vertices**
- They describe **how strongly each vertex is connected to the other vertices**



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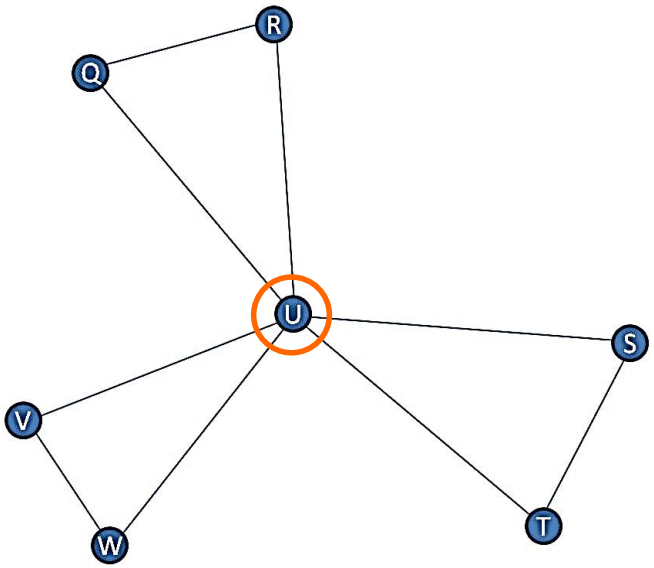

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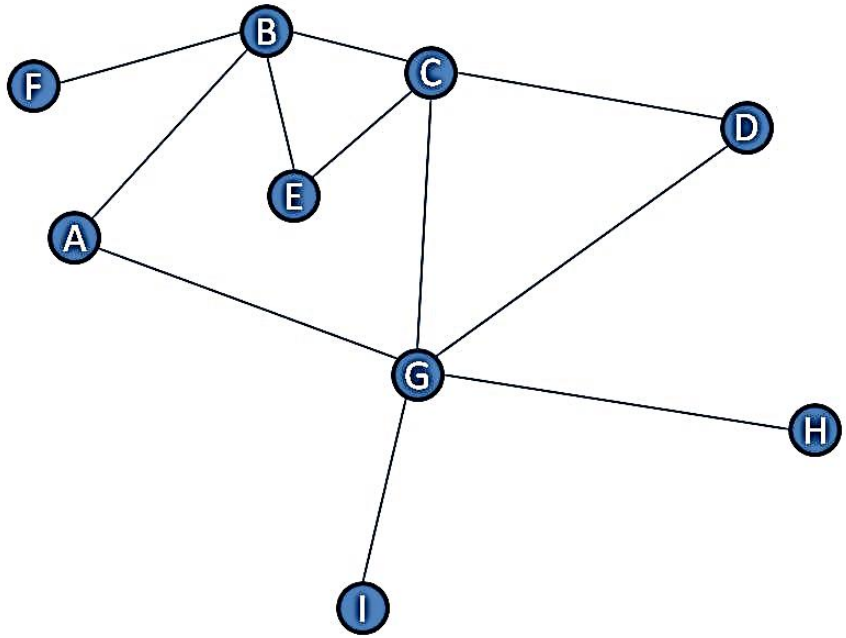
Gould Index of Accessibility

- Now consider a more realistic graph
- In this case, would **G** be the trade center, or would it be **C**?



Gould Index of Accessibility

- Now consider a more realistic graph
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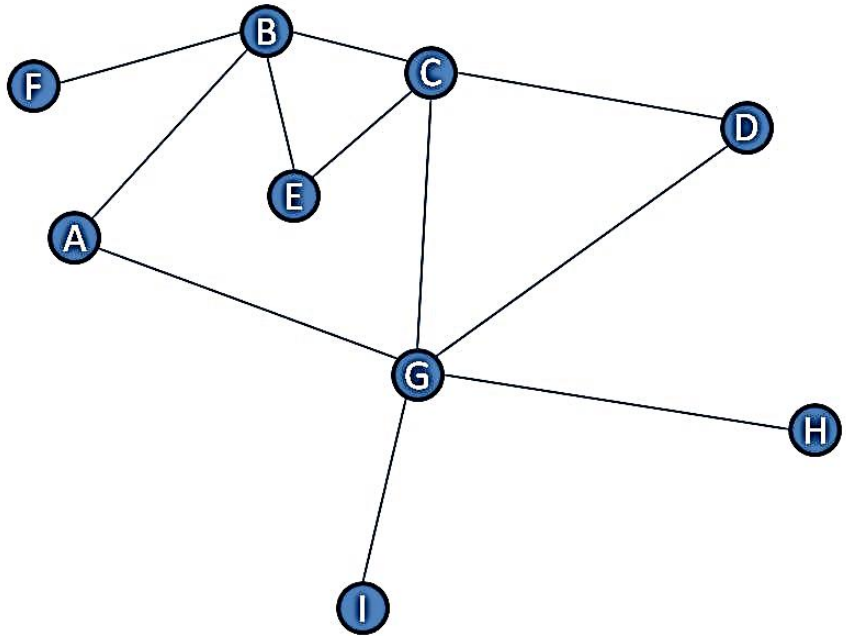
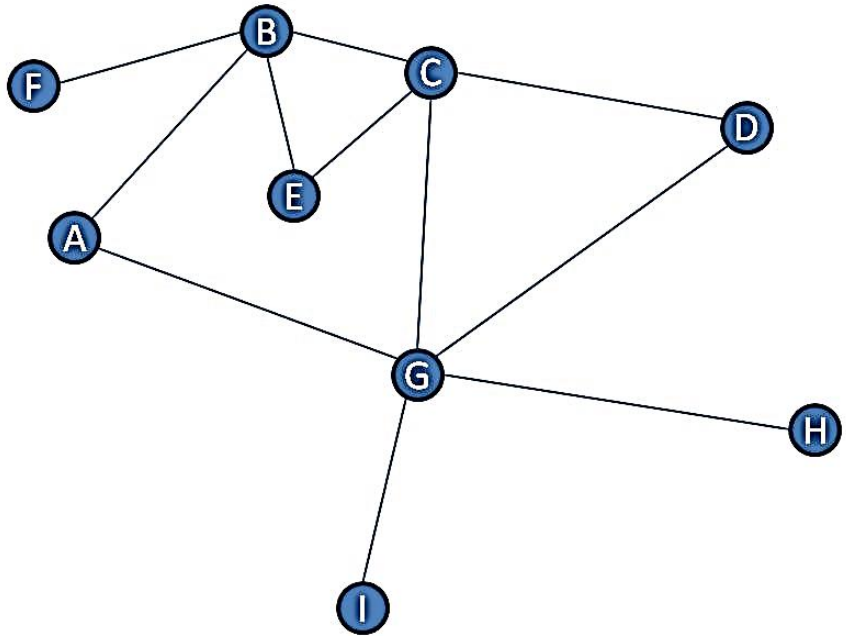
- The eigenvalues are:
  (4.01, -1.37, 1.71, 1, -0.37, -0.56, 2.58, 1, 1)



Gould Index of Accessibility

- The eigenvector associated with the largest eigenvalue is (0.2941, 0.4097, 0.5023, 0.3249, 0.3026, 0.1359, 0.4770, 0.1583, 0.1583)

- Normalizing we have (A, B, C, D, E, F, G, H, I) = (0.1064, 0.1482, 0.1818, 0.1176, 0.1095, 0.0492, 0.1726, 0.0573, 0.0573)



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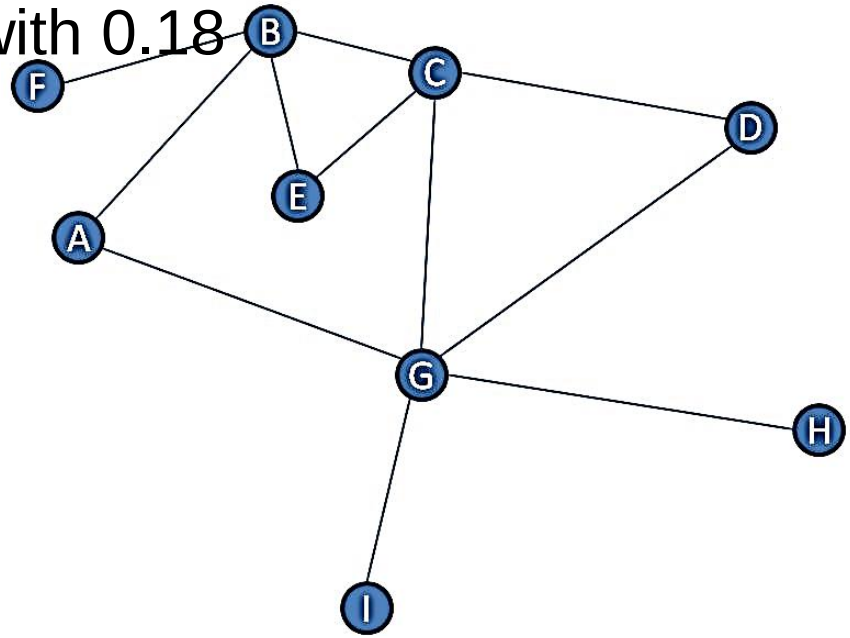
- **C** has the largest Gould Index with 0.18
- **G** comes second with 0.17



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- **C** has the largest Gould Index with 0.18
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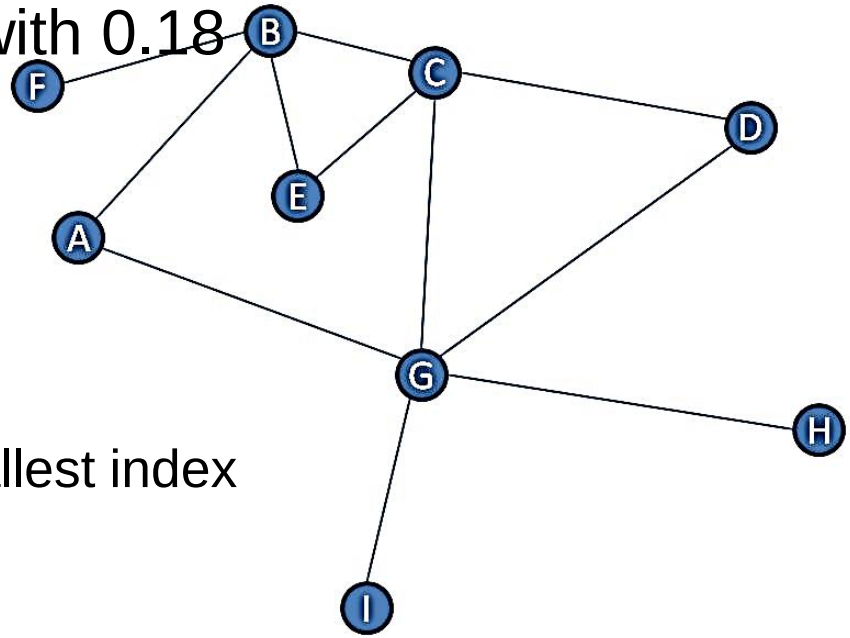
- Note that:
  - **F** has the smallest index
  - **I** and **H** have the same second smallest index



Algebraic Connectivity of graphs

- The importance of the **algebraic connectivity** of a graph is due to the fact that it is a good parameter to measure, to a certain extent, how well a graph is connected

- The algebraic connectivity is for
  - application on trees,
  - application on hard problems in graph theory (the expanding properties of graphs, weighted graphs, absolute algebraic connectivity, isoperimetric number, genus and other invariants of a graph)
  - the study of the asymptotic behavior for random graphs;
  - applications on combinatorial optimization problems (the maximum cut problem and the traveling salesman problem)
  - the theory of elasticity
  - the correspondence between continuous and discrete mathematics

Algebraic Connectivity of graphs

- Based on theory of **Fiedler** (1970s)

- We define the Laplacian matrix **L(G)** of the graph G(N,E) as:
  - L(G) (i,i) = degree of node I (number of incident edges)
  - L(G) (i,j) = -1 if i != j and there is an edge (i,j)
  - L(G) (i,j) = 0 otherwise

- We have **L(G)=D-A** where *D is the diagonal matrix* of node degrees and *A is the adjacency matrix* of graph G(N,E)

Algebraic Connectivity of Graphs

- L(G) is **symmetric:**
  - the eigenvalues of L(G) are **real**
  - the eigenvectors are **real** and **orthogonal**

- Further, the **eigenvalues** of L(G) are **nonnegative**:
  - $0 = \lambda_1 <= \lambda_2 <= \ldots <= \lambda_n$

- The number of connected components of G is equal to the **number of λ_i equal to 0**

- G is connected **if and only if λ_2 != 0**

Algebraic Connectivity of Graphs

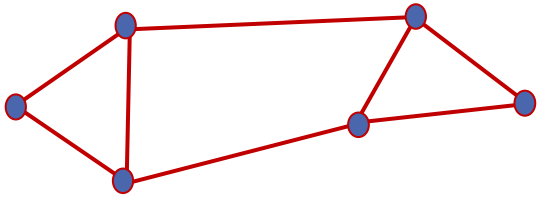
- The **second smallest eigenvalue** λ_2 of L(G) is called **algebraic connectivity** of G

- The eigenvector associated with the algebraic connectivity has been named the ***Fiedler vector***

- The Fiedler vector can be used to partition a graph

Graph Partitioning

Spectral Bi-partitioning Algorithm



- Build the Laplacian matrix L of the graph
- Find the second smallest eigenvalue and the corresponding eigenvector
- Map vertices to corresponding components of the Fiedler vector
- Grouping
  - Sort components in the Fiedler vector
  - Identify clusters by splitting the sorted vector in two

Graph Partitioning

How to choose a splitting point?

- Split at 0, that is positive and negative values
- Mean value
- Median value

- Partitioning a graph into *k* clusters can be done by **Recursive bi-partitioning** (Hagen et al.,'91)

- Recursively apply bi-partitioning algorithm in a hierarchical divisive manner
- Example: Image Segmentation
  - Uses 2nd (smallest) eigenvector to define optimal cut
  - Recursively generates two clusters with each cut

Markov chain

- http://webspace.ship.edu/deensley/m318/ppt/Section_49.pdf

- A vector with nonnegative entries that add up to 1 is called a **probability vector**

- A **stochastic matrix** is a square matrix whose columns are probability vectors

- A **Markov chain** is a sequence of probability vectors x_0, x_1, … together with a stochastic matrix P, such that

$$x_1 = Px_0 \qquad x_2 = Px_1 \qquad x_3 = Px_2 \qquad \ldots$$

Markov chain

Example

- Consider the following model of population movement between a city and the suburbs:
  - each year 5% of city dwellers move the suburbs and
  - 3% of suburbanites move to the city

- If in 2001 58.2% of the population lived in the city and 41.8% lived in the suburbs, what is the ***population distribution 20 years later***?

Markov chain

Example

- Consider matrix $M = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix}$ and vector $x_0 = \begin{pmatrix} 0.582 \\ 0.418 \end{pmatrix}$

  to represent the population distribution in 2001

- In this case $x_1 = Mx_0 = \begin{pmatrix} 0.565 \\ 0.435 \end{pmatrix}$ gives the population distribution in 2002
- In general $x_n = M^n x_0$ **gives the population distribution in *n* years after 2001**

Markov chain

- The difference between x_i and x_{i+1} gets smaller every step

- Let P be stochastic matrix
- A **steady-state vector** (also called **equilibrium vector**) is a probability vector x such that $\boldsymbol{Px=x}$
- If state x is achieved, the system stays there

- **A nonzero steady-state vector** is in fact an **eigenvector of eigenvalue 1 of P** (since it satisfies $Px=x$)

Markov chain

- In our example, the **eigenvalues** are $\lambda_1 = 1$ and $\lambda_2 = 0.92$, and the corresponding **eigenvectors** are

$$v_1 = \begin{pmatrix} -0.514496 \\ -0.857493 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -0.707107 \\ 0.707107 \end{pmatrix}$$

- So we form matrix P using eigenvectors and matrix D using eigenvalues

$$P = \begin{pmatrix} 0.514496 & -0.707107 \\ -0.85749 & 0.707107 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.92 \end{pmatrix}$$

Markov chain

- We can apply the diagonalization and obtain $M = PDP^{-1}$
- Now $M \cdot M = PDP^{-1}PDP^{-1} = PD^2P^{-1}$
- And, in general $M^k = PD^kP^{-1}$
- **Having** $M^k = P\begin{pmatrix} 1 & 0 \\ 0 & (0.92)^k \end{pmatrix}P^{-1}$
- **It follows** $\lim_{k\to\infty} M^k = P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P^{-1} = \begin{pmatrix} 0.375 & 0.375 \\ 0.625 & 0.625 \end{pmatrix}$

Markov chain

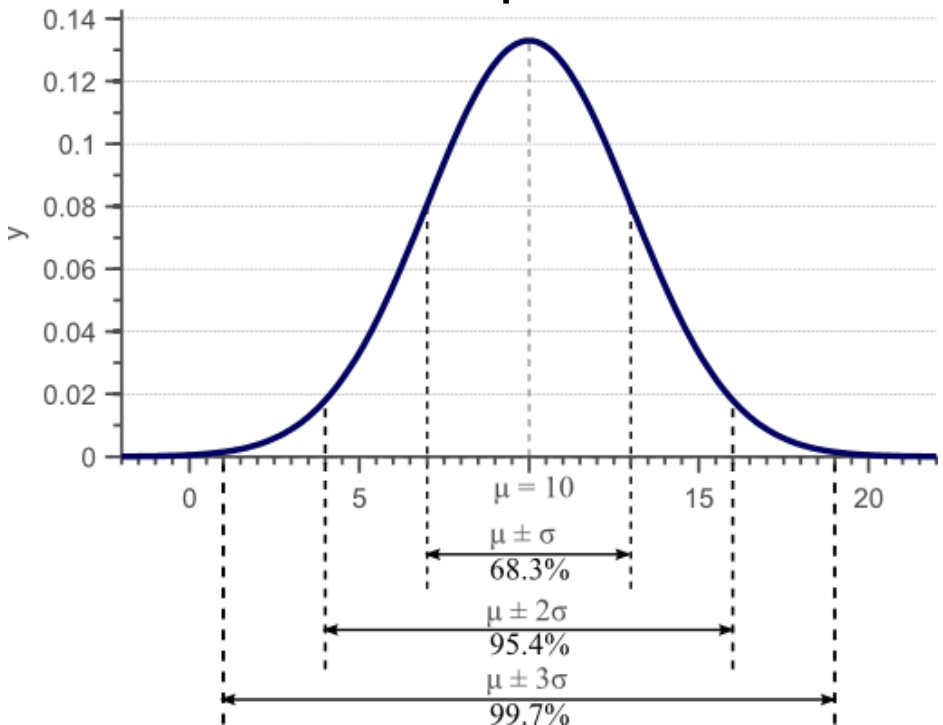
- For any initial stochastic vector x we will have

$$\lim_{k\to\infty} M^k\, x = \begin{pmatrix} 0.375 & 0.375 \\ 0.625 & 0.625 \end{pmatrix} x =$$

$$\begin{pmatrix} 0.375\, x_1 + 0.375\, x_2 \\ 0.625\, x_1 + 0.625\, x_2 \end{pmatrix} = \begin{pmatrix} 0.375 \\ 0.625 \end{pmatrix}$$

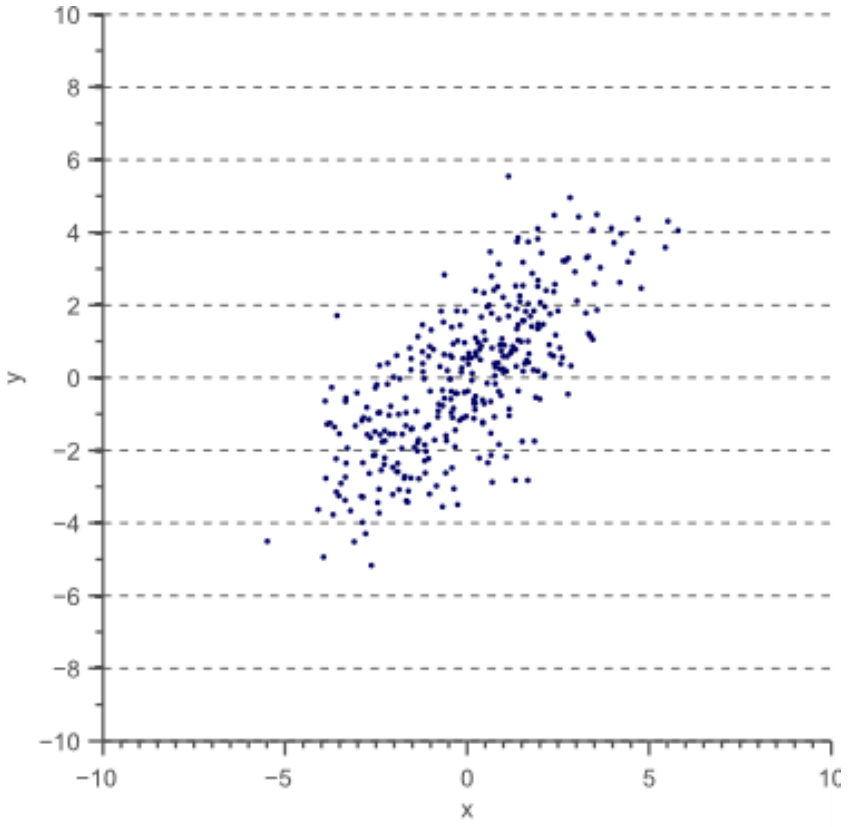
The covariance matrix

- https://www.visiondummy.com/2014/04/geometric-interpretation-covariance-matrix/
- Given a set of data, the standard deviation (square root of the variance) provides a measure of how much the data is spread across the feature space



The covariance matrix

- However, variance can only be used to explain the spread of the data in the directions parallel to the axes of the space
- We could calculate the variance σ(x,x) in the x-direction and the variance σ(y,y) in the y-direction
- However, the horizontal spread and the vertical spread of the data does not explain the clear diagonal correlation

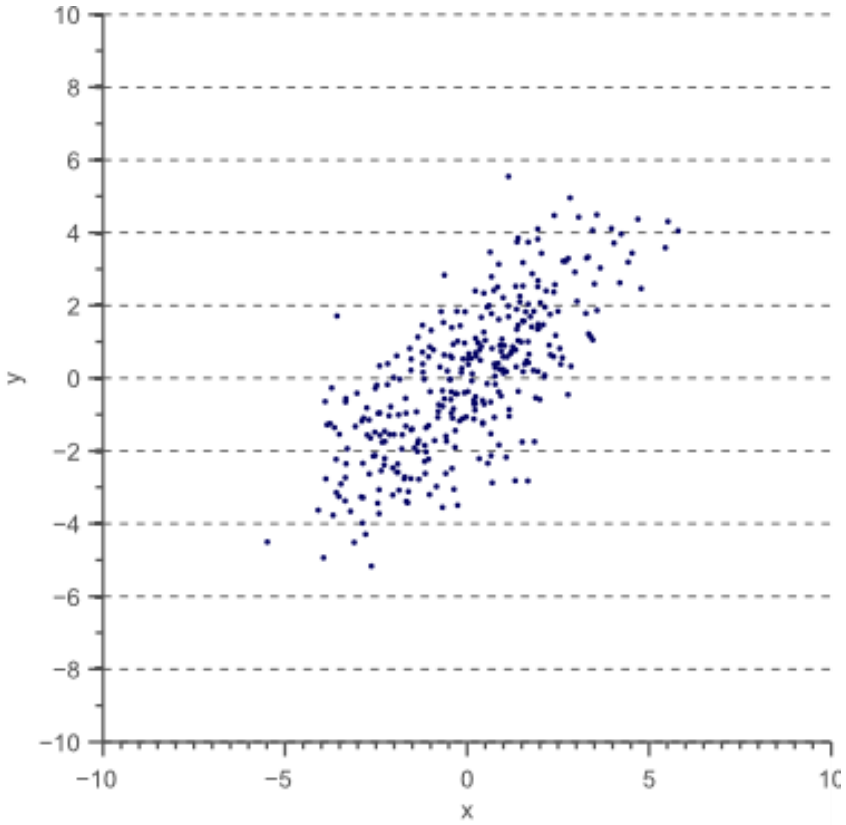


The covariance matrix

- The correlation between x-value data and y-value data can be captured by extending the notion of variance to the ***covariance*** of the data:
- $\Sigma(x,y) = E[(x-(E(x))(y- E(y))]$

- For 2D data, we thus obtain $\sigma(x,x)$, $\sigma(y,y)$, $\sigma(x,y)$ and $\sigma(y,x)$
- These four values can be summarized in the **covariance matrix**, $\sum$:

$\sigma(x,x)$	$\sigma(x,y)$
$\sigma(y,x)$	$\sigma(y,y)$

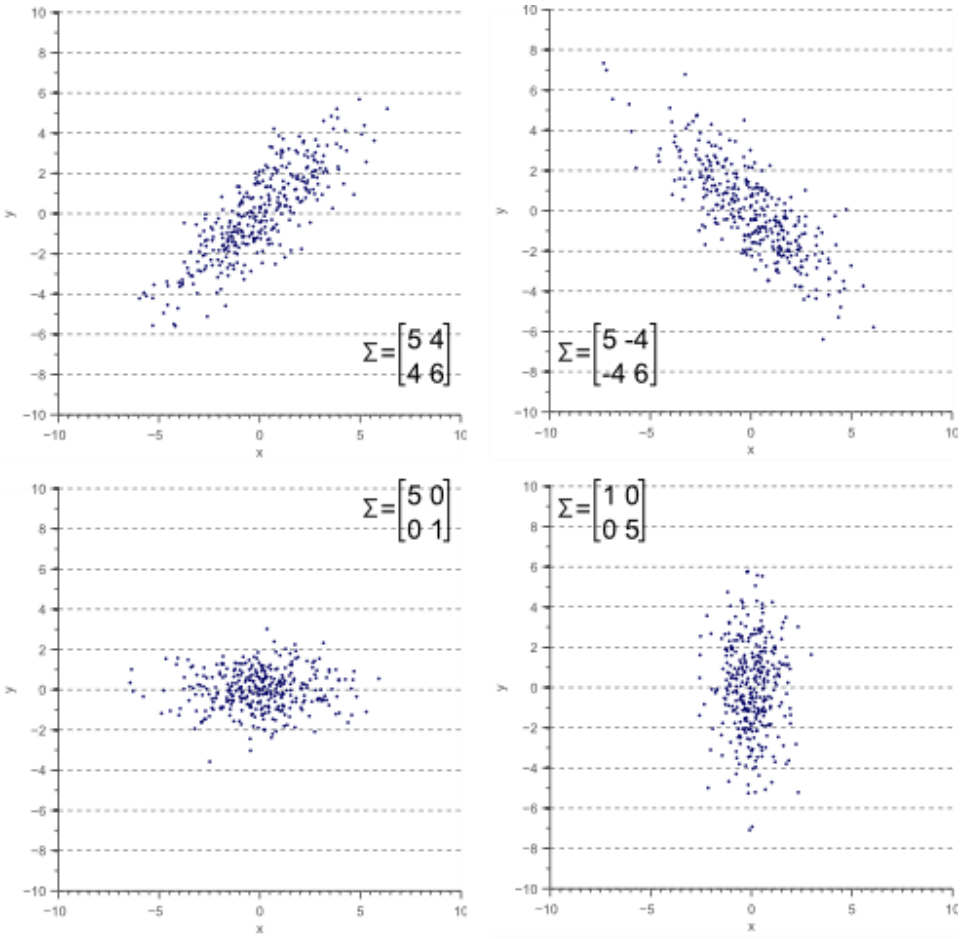


The covariance matrix

- If x is positively correlated with y, y is also positively correlated with x. In other words, we can state that:

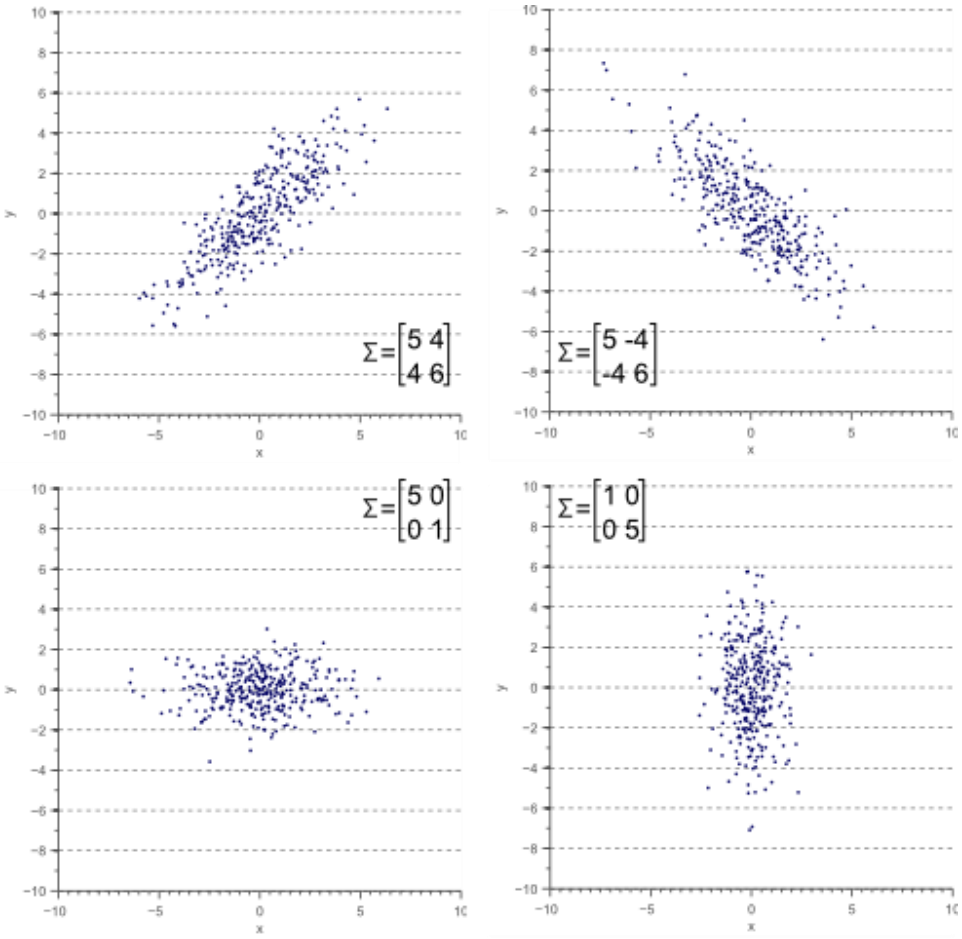
$$\sigma(x,y) = \sigma(y,x)$$

- Therefore, the **covariance matrix** is always a **symmetric matrix** with the variances on its diagonal and the covariances off-diagonal



The covariance matrix

- The covariance matrix defines both the spread (**variance**), and the orientation (**covariance**) of data
- To represent the covariance matrix with a vector and its magnitude, we have to find the vector that points into the direction of the largest spread of the data, and whose magnitude equals the spread (variance) in this direction



The covariance matrix

- The largest eigenvector of the covariance matrix always points into the direction of the largest variance of the data, and the its magnitude equals the corresponding eigenvalue
- The second largest eigenvector is always orthogonal to the largest eigenvector, and points into the direction of the second largest spread of the data