Linear Systems – Part 1

Intensive Computation

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Systems of linear equations

- The **linear system**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 \ldots + a_{2n}x_n &= b_2 \\
\ldots \\
a_{n1}x_1 + a_{n2}x_2 \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

for unknown $x_1,\ldots,x_n;$ and a_{ij}, b_i constants for $i, j = 1,2,\ldots, n$

- In matrix form $A\,x = b$

- Given $A\,x = b$:
  - Is there a solution?
  - Is the solution unique?

Systems of linear equations

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\end{aligned}
$$

for unknown $x_1,\ldots,x_n$; and a_{ij}, b_i constants for $i, j = 1,2,\ldots, n$

- In matrix form $A\,x = b$

- Given $A\,x = b$:
  - Is there a solution?
  - Is the solution unique?

Yes, if A **square** and **nonsingular** (determinant ≠ 0)

Systems of linear equations

- We consider A **square** and **nonsingular**
- If the matrix A is **diagonal**

$$\begin{bmatrix} a_{11} & 0 & & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ . & . & & & . \\ . & . & & & . \\ 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

- The solution is $\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = x$ where $x_i = \dfrac{b_i}{a_{ii}}$

Systems of linear equations

- If **A** is an **upper triangular** matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ . & . & & . \\ . & . & & . \\ 0 \dots\dots\dots\dots\dots\dots & 0 & & a_{nn} \end{bmatrix}$$

- The solution is $\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = x$ where

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_i = \frac{1}{a_{ii}}(b_i - \sum_{k=i+1}^{n} a_{ik} x_k) \quad i = n-1,\dots,1$$

Backward substitution

Systems of linear equations

- If A is a **lower triangular** matrix

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22}\ 0 & \dots & 0 \\ . & . & & . \\ . & . & & . \\ a_{n1} & \dots\dots\dots & & a_{nn} \end{bmatrix}$$

- The solution is $\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = x$ where

$$\boxed{\begin{aligned} x_1 &= \frac{b_1}{a_{11}} \\ x_i &= \frac{1}{a_{ii}}\left(b_i - \sum_{k=1}^{i-1} a_{ik} x_k\right) \quad i = 2,\dots,n \end{aligned}}$$

Forward substitution

Systems of linear equations

- If A is **ortogonal**, that is $A=A^T$, the solution is $x=A^Tb$

- Let us consider

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & . \\ . & . & & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \mathbf{A} \qquad \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix} = \mathbf{b}$$

Then $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ . & . & & . & : & . \\ . & . & & . & : & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} & : & b_n \end{bmatrix} = [\mathbf{A}, \mathbf{b}]$ is **the augmented matrix**

Methods for solving linear equations

- **Direct methods**: find the ***exact*** solution in a finite number of steps
- **Iterative methods**: produce a sequence of approximate solutions hopefully converging to the exact solution

Gaussian Elimination Method

Gaussian Elimination Method for solving $A\,x = b$

- A **direct** method

- The idea is:
  - To transform the system such that the matrix ***A is transformed in a triangular matrix***
  - The system can be solved by a **backward substitution** process

- The method:
  - Provides an exact result (***ignoring roundoff***)
  - Is *computationally expensive*

Gaussian Elimination Method

Gaussian Elimination Method for solving $A\,x = b$

To transform the matrix A, three **row operations** can be performed on the rows of a matrix ***without altering the considered system***:

- **Operation 1**: Swap the positions of two rows
- **Operation 2**: Multiply a row by a nonzero scalar
- **Operation 3**: Add to one row a scalar multiple of another

Gaussian Elimination

Consider a 3x3 example

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

$$\begin{aligned} 6x_1 + 10x_2 + 4x_3 &= 2 \\ 5x_1 + 9x_2 + 6x_3 &= 11 \\ 17x_1 + 26x_2 + 21x_3 &= 49 \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 10 & 4 \\ 5 & 9 & 6 \\ 17 & 26 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 49 \end{bmatrix}$$

Gaussian Elimination

Use Equation 1 to **eliminate x_1** from Equation 2 and 3

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

This is done by ***subtracting from Equation i*** Equation 1

multiplied for $\boxed{m_{i1} = \frac{a_{i1}}{a_{11}}}$ that are the ***multipliers***

Pivot

That is:

$$(a_{22} - \frac{a_{21}}{a_{11}} a_{12}) x_2 + (a_{23} - \frac{a_{21}}{a_{11}} a_{13}) x_3 = b_2 - \frac{a_{21}}{a_{11}} b_1$$

$$(a_{32} - \frac{a_{31}}{a_{11}} a_{12}) x_2 + (a_{33} - \frac{a_{31}}{a_{11}} a_{13}) x_3 = b_3 - \frac{a_{31}}{a_{11}} b_1$$

Gaussian Elimination

Use Equation 1 to **eliminate x_1** from Equation 2 and 3

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

This is done by ***subtracting from Equation i*** Equation 1 multiplied for $\boxed{m_{i1} = \frac{a_{i1}}{a_{11}}}$ that are the ***multipliers***

That is:

$$(a_{22} - m_{21}a_{12})x_2 + (a_{23} - m_{21}a_{13})x_3 = b_2 - m_{21}b_1$$

$$(a_{32} - m_{31}a_{12})x_2 + (a_{33} - m_{31}a_{13})x_3 = b_3 - m_{31}b_1$$

Gaussian Elimination

Eliminate x_1 from Equation 2 and 3
Matrix form

Pivot **MULTIPLIERS**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right) & \left(a_{23} - \frac{a_{21}}{a_{11}} a_{13}\right) \\ 0 & \left(a_{32} - \frac{a_{31}}{a_{11}} a_{12}\right) & \left(a_{33} - \frac{a_{31}}{a_{11}} a_{13}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - \frac{a_{21}}{a_{11}} b_1 \\ b_3 - \frac{a_{31}}{a_{11}} b_1 \end{bmatrix}$$

Gaussian Elimination

Eliminate x_1 from Equation 2 and 3
Matrix form

Pivot **MULTIPLIERS**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & (a_{22} - m_{21}a_{12}) & (a_{23} - m_{21}a_{13}) \\ 0 & (a_{32} - m_{31}a_{12}) & (a_{33} - m_{31}a_{13}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - m_{21}b_1 \\ b_3 - m_{31}b_1 \end{bmatrix}$$

Gaussian Elimination

Eliminate x_1 from Equation 2 and 3

Pivot a_{11}

Multipliers

$$m_{21} = \frac{a_{21}}{a_{11}}$$

$$m_{31} = \frac{a_{31}}{a_{11}}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} \end{bmatrix}$$

$$A^{(1)} x = b^{(1)}$$

Gaussian Elimination

Eliminate x_1 from Equation 2 and 3

Pivot a_{11}

Multipliers

$$m_{21} = \frac{a_{21}}{a_{11}}$$

$$m_{31} = \frac{a_{31}}{a_{11}}$$

$$\begin{bmatrix} 6 & 10 & 4 \\ 0 & \frac{2}{3} & \frac{8}{3} \\ 0 & -\frac{7}{3} & \frac{29}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ \frac{28}{3} \\ \frac{130}{3} \end{bmatrix}$$

Gaussian Elimination

Eliminate x_2 from Equation 3

Matrix form

Pivot

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & \left(a_{33}^{(1)} - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} a_{23}^{(1)}\right) \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} a_{33}^{(1)} - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} b_2^{(1)} \end{bmatrix}
$$

Multiplier

Gaussian Elimination

GE yields triangular system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \end{bmatrix}$$

Altered During GE

$$A^{(2)} x = b^{(2)}$$

Gaussian Elimination

GE yields triangular system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \end{bmatrix}$$

Altered During GE

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_3 = \frac{y_3}{u_{33}}$$

Backward substitution

$$x_2 = \frac{y_2 - u_{23}x_3}{u_{22}}$$

$$x_1 = \frac{y_1 - u_{12}x_2 - u_{13}x_3}{u_{11}}$$

Gaussian Elimination

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - (\frac{a_{21}}{a_{11}})\ b_1 \\ b_3 - (\frac{a_{31}}{a_{11}})\ b_1 - (\frac{a_{32}^{(1)}}{a_{22}^{(1)}})\ b_2^{(1)} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a_{32}^{(1)}}{a_{22}^{(1)}} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Gaussian Elimination

In general

- The ***k*th stage** of the elimination zeros the elements below the pivot element $a_{kk}^{(k)}$ in the *k*th column of $A^{(k)}$ according to the operations

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - m_{ik} a_{kj}^{(k)} \qquad i, j = k+1 : n$$

$$b_i^{(k+1)} = b_i^{(k)} - m_{ik} b_k^{(k)} \qquad i = k+1 : n$$

- where the quantities $m_{ik} = \dfrac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \quad i = k+1 : n$ are called the **multipliers** and $a_{kk}^{(k)}$ is called the **pivot**

Gaussian Elimination

- $A\,x = b$ ➔ $LU\,x = b$ ➔ **LU factorization**
- $Ly = b$ *L:* lower triangular
  multipliers and 1s on diagonal
- $U\,x = y$ *U:* upper triangular

Gaussian Elimination

Solve $A\,x = b$

Step 1 $A = LU$ ➔ **LU factorization**

Step 2 $Ly = b$ ➔ Solve by forward substitution

Step 3 $U\,x = y$ ➔ Solve by backward substitution

Gaussian Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{a_{21}}{a_{11}} & 1 & 0 \\ \dfrac{a_{21}}{a_{11}} & \dfrac{a_{32}^{(1)}}{a_{22}^{(1)}} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}$$

Gaussian Elimination

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{a_{21}}{a_{11}} & 1 & 0 \\ \dfrac{a_{21}}{a_{11}} & \dfrac{a_{32}^{(1)}}{a_{22}^{(1)}} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

LU decomposition - Algorithm

For k = 1 to n-1 ***Consider the augmented matrix Ab***

$$m(k+1:n,k) = a(k+1:n,k) / a(k,k)$$

Multiplier

Pivot

$$a(k+1:n,k+1:n+1) = a(k+1:n,k+1:n+1) + \\ - m(k+1:n,k) * a(k,k+1:n+1)$$

Note that:

- Elements of matrix A are **not** zeroed
- Multipliers can be stored in matrix A (instead of zeroing the elementsof A)

LU decomposition – Algorithm cost

For k = 1 to n-1 ***Consider the augmented matrix Ab***

$$m(k+1:n,k) = \underbrace{a(k+1:n,k) \big/ a(k,k)}_{\text{(n-k) divisions}}$$

$$a(k+1:n,k+1:n+1) = a(k+1:n,k+1:n+1) + \underbrace{-m(k+1:n,k) * a(k,k+1:n+1)}_{\text{(n-k)(n-k+1) multiplications}}$$

Cost:

- **(n-k)(n-k+2)** multiplication and division operations

LU decomposition – Algorithm cost

For k = 1 to n-1 ***Consider the augmented matrix Ab***

$$\underbrace{m(k+1:n,k) = a(k+1:n,k) / a(k,k)}_{\text{(n-k) divisions}}$$

$$a(k+1:n,k+1:n+1) = a(k+1:n,k+1:n+1) + \underbrace{-m(k+1:n,k) * a(k,k+1:n+1)}_{\text{(n-k)(n-k+1) multiplications}}$$

Cost:

- **(n-k)(n-k+2)** multiplication and division operations
- **n-1 iterations**

$$\rightarrow \sum_{k=1}^{n-1} (n-k)(n-k+2) = O(n^3)$$

LU decomposition – Algorithm cost

Further we have:

- Multiplication and division operations for solving the triangular systems: ***$O(n^2)$***
- Addition and subtraction operations: ***$O(n^3)$***
- *Note that addition/subtraction operations have a lower cost with respect to multiplication/division operations, that is $O(l)$ vs $O(l^2)$ where l is the lenght of operands*
- *Anyway usually we consider the number of floating point operations FLOPs*

Limitation to this approach

Gaussian Elimination method **may fail**:

- **Null pivots**: we have a division by zero during formation of the multipliers if a
- **Small pivots** (Round-off error)

Both can be solved with **partial pivoting**

Limitation to this approach

- If at iteration *k* $a_{kk}^{(k)} = 0$ we cannot form the multipliers

$$m_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \quad i = k+1 : n$$

- A simple way to overcome this problem is the following:

  If $a_{kk}^{(k)} = 0$

  Find $a_{ik}^{(k)} \neq 0$ with $i > k$

  Swap row *i* with row *k*

Limitation to this approach

Two Important Theorems

- Partial pivoting (swapping rows) always succeeds if M is non singular

- LU factorization applied to a diagonally dominant matrix will never produce a zero pivot

Limitation to this approach

- In practical computation, **also *small pivots can give problems***

- **Small pivots** can lead to **large multipliers m_{ik}**

- If m_{ik} is large then there is a possible loss of significance in the subtraction $\boldsymbol{a_{ij}^{(k)} - m_{ik} a_{kj}^{(k)}}$, with low-order digits of $a_{ij}^{(k)}$ being lost

- Losing these digits could correspond to making a relatively large ***change to the original matrix A***

Pivoting and Numerical Stability

- Three different **pivoting strategies** to avoid instability

- All three strategies ensure that the multipliers are nicely bounded: $|m_{ik}| < 1 \quad i = k+1 : n$

Partial pivoting

At *k*th stage, the *k*th and *r*th rows are interchanged, where

$$\left|a_{rk}^{(k)}\right| = \max_{k \le i \le n} \left|a_{ik}^{(k)}\right|$$

- **Pivot of maximal magnitude over the the pivot column** is selected as pivot

Pivoting and Numerical Stability

Complete pivoting

At *k*th stage rows *k* and *r* and columns *k* and *s* are swapped

$$\left|a_{rs}^{(k)}\right| = \max_{k \le i,\, j \le n} \left|a_{ij}^{(k)}\right|$$

- **Pivot of maximal magnitude over the whole submatrix**

Rook pivoting

At *k*th stage, rows *k* and *r* and columns *k* and *s* are swapped

$$\left|a_{rs}^{(k)}\right| = \max_{k \le i \le n} \left|a_{is}^{(k)}\right| = \max_{k \le j \le n} \left|a_{rj}^{(k)}\right|$$

- **Pivot of maximal magnitude in both its column and its row**

Pivoting and Numerical Stability

Consider the system

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

After GE

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 0 & 12.5-(1.25\cdot 10^{5}) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 6.25 \\ 75-(6.25\cdot 10^{5}) \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5digits}=\begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

Pivoting and Numerical Stability

Consider the system

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

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$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5digits} = \begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{3digits} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}}$$

Pivoting and Numerical Stability

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

Swap rows

$$\begin{bmatrix} 12.5 & 12.5 \\ 1.25\cdot 10^{-4} & 1.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$

GE

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 - 12.5\cdot 10^{-5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 - 75\cdot 10^{-5} \end{bmatrix}$$

Rounded to 5 digits

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5digits} = \begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

Pivoting and Numerical Stability

$$\begin{bmatrix} 1.25\cdot10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

swap

$$\begin{bmatrix} 12.5 & 12.5 \\ 1.25\cdot10^{-4} & 1.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$

GE

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 - 12.5\cdot10^{-5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 - 75\cdot10^{-5} \end{bmatrix}$$

Rounded to 3 digits

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{3digits} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5digits} = \begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

Fill-in Problem

- If we have a **sparse matrix**, Gaussian elimination could **destroy its sparsity**
- Consider the matrix

$$A = \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & 1 & 0 & 0 & 0 \\ \frac{1}{10} & 0 & 1 & 0 & 0 \\ \frac{1}{10} & 0 & 0 & 1 & 0 \\ \frac{1}{10} & 0 & 0 & 0 & 1 \end{bmatrix}$$

Fill-in Problem

- Example

$$A=\begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & 1 & 0 & 0 & 0 \\ \frac{1}{10} & 0 & 1 & 0 & 0 \\ \frac{1}{10} & 0 & 0 & 1 & 0 \\ \frac{1}{10} & 0 & 0 & 0 & 1 \end{bmatrix}$$

First step of Gaussian Elimination

$$\begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ 0 & \frac{99}{100} & -\frac{1}{100} & -\frac{1}{100} & -\frac{1}{100} \\ 0 & -\frac{1}{100} & \frac{99}{100} & -\frac{1}{100} & -\frac{1}{100} \\ 0 & -\frac{1}{100} & -\frac{1}{100} & \frac{99}{100} & -\frac{1}{100} \\ 0 & -\frac{1}{100} & -\frac{1}{100} & -\frac{1}{100} & \frac{99}{100} \end{bmatrix}$$

Fill-in Problem

- Example

$$A = \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & 1 & 0 & 0 & 0 \\ \frac{1}{10} & 0 & 1 & 0 & 0 \\ \frac{1}{10} & 0 & 0 & 1 & 0 \\ \frac{1}{10} & 0 & 0 & 0 & 1 \end{bmatrix} \quad \overset{\text{LU factorization}}{\Longrightarrow} \quad \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{99}{100} & -\frac{1}{100} & -\frac{1}{100} & -\frac{1}{100} \\ \frac{1}{10} & -\frac{1}{99} & \frac{98}{99} & -\frac{1}{99} & -\frac{1}{99} \\ \frac{1}{10} & -\frac{1}{99} & -\frac{1}{98} & \frac{97}{98} & -\frac{1}{98} \\ \frac{1}{10} & -\frac{1}{99} & -\frac{1}{98} & -\frac{1}{97} & \frac{96}{97} \end{bmatrix}$$

Fill-in Problem

- Reorder the matrix by swapping the first and last row and then the first and last column

$$Pr = Pc = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad PrAPc = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{10} \\ 0 & 1 & 0 & 0 & \frac{1}{10} \\ 0 & 0 & 1 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 1 & \frac{1}{10} \\ 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \end{bmatrix}$$

Fill-in Problem

- After the LU factorization there is not fill-in

$$PrAPc = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{10} \\ 0 & 1 & 0 & 0 & \frac{1}{10} \\ 0 & 0 & 1 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 1 & \frac{1}{10} \\ 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \end{bmatrix} \xrightarrow{\text{LU factorization}} \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{10} \\ 0 & 1 & 0 & 0 & \frac{1}{10} \\ 0 & 0 & 1 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 1 & \frac{1}{10} \\ 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & -\frac{3}{100} \end{bmatrix}$$