

Linear Systems – Part 1

Intensive Computation

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Systems of linear equations

- The **linear system**

$$a_{11}x_1 + a_{12}x_2 \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 \dots + a_{2n}x_n = b_2$$

...

$$a_{n1}x_1 + a_{n2}x_2 \dots + a_{nn}x_n = b_n$$

for unknown x_1, \dots, x_n ; and a_{ij}, b_i constants for $i, j = 1, 2, \dots, n$

- In matrix form $Ax = b$
- Given $Ax = b$:
 - Is there a solution?
 - Is the solution unique?

Systems of linear equations

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for unknown x_1, \dots, x_n ; and a_{ij}, b_i constants for $i, j = 1, 2, \dots, n$

- In matrix form $Ax = b$
- Given $Ax = b$:
 - Is there a solution?
 - Is the solution unique?

Yes, if A **square** and **nonsingular** (determinant $\neq 0$)

Systems of linear equations

- We consider A **square** and **nonsingular**
- If the matrix A is **diagonal**

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

- The solution is $\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = x$ where $x_i = \frac{b_i}{a_{ii}}$

Systems of linear equations

- If **A** is an **upper triangular** matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

- The solution is $[x_1 \ x_2 \ \dots \ x_n] = x$ where

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_i = \frac{1}{a_{ii}} \left(b_i - \sum_{k=i+1}^n a_{ik} x_k \right) \quad i = n-1, \dots, 1$$

**Backward
substitution**

Systems of linear equations

- If A is a **lower triangular** matrix

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \dots & \dots & \dots & a_{nn} \end{bmatrix}$$

- The solution is $[x_1 \ x_2 \ \dots \ x_n] = x$ where

$$x_1 = \frac{b_1}{a_{11}}$$

$$x_i = \frac{1}{a_{ii}} \left(b_i - \sum_{k=1}^{i-1} a_{ik} x_k \right) \quad i = 2, \dots, n$$

**Forward
substitution**

Systems of linear equations

- If A is **ortogonal**, that is $A=A^T$, the solution is $x=A^Tb$
- Let us consider

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \mathbf{A} \quad \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{bmatrix} = \mathbf{b}$$

Then

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ \cdot & \cdot & & \cdot & : & \cdot \\ \cdot & \cdot & & \cdot & : & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} & : & b_n \end{bmatrix} = [\mathbf{A}, \mathbf{b}] \quad \text{is the augmented matrix}$$

Methods for solving linear equations

- **Direct methods**: find the ***exact*** solution in a finite number of steps
- **Iterative methods**: produce a sequence of approximate solutions hopefully converging to the exact solution

Gaussian Elimination Method

Gaussian Elimination Method for solving $Ax = b$

- A **direct** method
- The idea is:
 - To transform the system such that the matrix ***A is transformed in a triangular matrix***
 - The system can be solved by a **backward substitution** process
- The method:
 - Provides an exact result (***ignoring roundoff***)
 - Is *computationally expensive*

Gaussian Elimination Method

Gaussian Elimination Method for solving $Ax = b$

To transform the matrix A , three **row operations** can be performed on the rows of a matrix *without altering the considered system*:

- **Operation 1:** Swap the positions of two rows
- **Operation 2:** Multiply a row by a nonzero scalar
- **Operation 3:** Add to one row a scalar multiple of another

Gaussian Elimination

Consider a 3x3 example

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$



$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$6x_1 + 10x_2 + 4x_3 = 2$$

$$5x_1 + 9x_2 + 6x_3 = 11$$

$$17x_1 + 26x_2 + 21x_3 = 49$$



$$\begin{bmatrix} 6 & 10 & 4 \\ 5 & 9 & 6 \\ 17 & 26 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 49 \end{bmatrix}$$

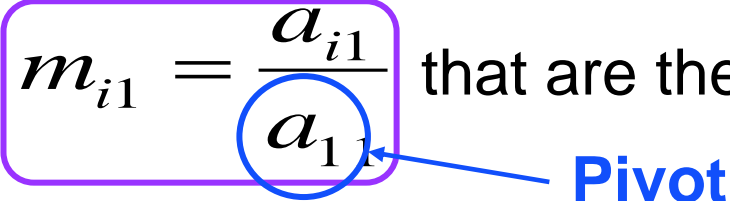
Gaussian Elimination

Use Equation 1 to **eliminate** x_1 from Equation 2 and 3

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

This is done by **subtracting from Equation i** Equation 1

multiplied for $m_{i1} = \frac{a_{i1}}{a_{11}}$ that are the **multipliers**



Pivot

That is:

$$\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right)x_2 + \left(a_{23} - \frac{a_{21}}{a_{11}} a_{13}\right)x_3 = b_2 - \frac{a_{21}}{a_{11}} b_1$$

$$\left(a_{32} - \frac{a_{31}}{a_{11}} a_{12}\right)x_2 + \left(a_{33} - \frac{a_{31}}{a_{11}} a_{13}\right)x_3 = b_3 - \frac{a_{31}}{a_{11}} b_1$$

Gaussian Elimination

Use Equation 1 to **eliminate** x_1 from Equation 2 and 3

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

This is done by **subtracting from Equation i** Equation 1

multiplied for $m_{i1} = \frac{a_{i1}}{a_{11}}$ that are the **multipliers**

That is:

$$(a_{22} - m_{21}a_{12})x_2 + (a_{23} - m_{21}a_{13})x_3 = b_2 - m_{21}b_1$$

$$(a_{32} - m_{31}a_{12})x_2 + (a_{33} - m_{31}a_{13})x_3 = b_3 - m_{31}b_1$$

Gaussian Elimination

Eliminate x_1 from Equation 2 and 3
Matrix form

Pivot

MULTIPLIERS

$$\begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix}
 \begin{matrix} a_{12} \\ (a_{22} - \frac{a_{21}}{a_{11}} a_{12}) \\ (a_{32} - \frac{a_{31}}{a_{11}} a_{12}) \end{matrix}
 \begin{matrix} a_{13} \\ (a_{23} - \frac{a_{21}}{a_{11}} a_{13}) \\ (a_{33} - \frac{a_{31}}{a_{11}} a_{13}) \end{matrix}
 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
 =
 \begin{bmatrix} b_1 \\ b_2 - \frac{a_{21}}{a_{11}} b_1 \\ b_3 - \frac{a_{31}}{a_{11}} b_1 \end{bmatrix}$$

Gaussian Elimination

Eliminate x_1 from Equation 2 and 3
Matrix form

Pivot

MULTIPLIERS

$$\begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix}
 \begin{matrix} a_{12} \\ (a_{22} - m_{21}a_{12}) \\ (a_{32} - m_{31}a_{12}) \end{matrix}
 \begin{matrix} a_{13} \\ (a_{23} - m_{21}a_{13}) \\ (a_{33} - m_{31}a_{13}) \end{matrix}
 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
 =
 \begin{bmatrix} b_1 \\ b_2 - m_{21}b_1 \\ b_3 - m_{31}b_1 \end{bmatrix}$$

Gaussian Elimination

Eliminate x_1 from Equation 2 and 3

Pivot a_{11}

Multipliers

$$m_{21} = \frac{a_{21}}{a_{11}}$$

$$m_{31} = \frac{a_{31}}{a_{11}}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} \end{bmatrix}$$



$$A^{(1)}x = b^{(1)}$$

Gaussian Elimination

Eliminate x_1 from Equation 2 and 3

Pivot a_{11}

Multipliers

$$m_{21} = \frac{a_{21}}{a_{11}}$$

$$m_{31} = \frac{a_{31}}{a_{11}}$$

$$\begin{bmatrix} 6 & 10 & 4 \\ 0 & \frac{2}{3} & \frac{8}{3} \\ 0 & -\frac{7}{3} & \frac{29}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ \frac{28}{3} \\ \frac{130}{3} \end{bmatrix}$$

Gaussian Elimination

Eliminate x_2 from Equation 3
Matrix form

$$\begin{bmatrix} a_{11} & a_{12} \text{ Pivot} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & (a_{33}^{(1)} - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} a_{23}^{(1)}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} a_{33}^{(1)} - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} b_2^{(1)} \end{bmatrix}$$

Multiplier

Gaussian Elimination

GE yields triangular system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \end{bmatrix}$$

**Altered
During
GE**



$$A^{(2)} x = b^{(2)}$$


Gaussian Elimination

GE yields triangular system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \end{bmatrix}$$

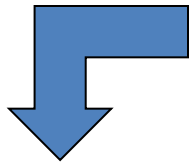
**Altered
During
GE**




$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$



$$x_3 = \frac{y_3}{u_{33}}$$

Backward substitution

$$x_2 = \frac{y_2 - u_{23}x_3}{u_{22}}$$

$$x_1 = \frac{y_1 - u_{12}x_2 - u_{13}x_3}{u_{11}}$$

Gaussian Elimination

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - \left(\frac{a_{21}}{a_{11}}\right) b_1 \\ b_3 - \left(\frac{a_{31}}{a_{11}}\right) b_1 - \left(\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right) b_2^{(1)} \end{bmatrix} \xrightarrow{\text{red arrow}} \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a_{32}^{(1)}}{a_{22}^{(1)}} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\xrightarrow{\text{red arrow}} \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Gaussian Elimination

In general

- The ***k*th stage** of the elimination zeros the elements below the pivot element $a_{kk}^{(k)}$ in the *k*th column of $A^{(k)}$ according to the operations

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - m_{ik} a_{kj}^{(k)} \quad i, j = k + 1 : n$$

$$b_i^{(k+1)} = b_i^{(k)} - m_{ik} b_k^{(k)} \quad i = k + 1 : n$$

- where the quantities $m_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \quad i = k + 1 : n$ are called the **multipliers** and $a_{kk}^{(k)}$ is called the **pivot**

Gaussian Elimination

- $Ax = b \rightarrow LUx = b \rightarrow$ **LU factorization**
- $Ly = b$ *L: lower triangular
multipliers and 1s on diagonal*
- $Ux = y$ *U: upper triangular*

Gaussian Elimination

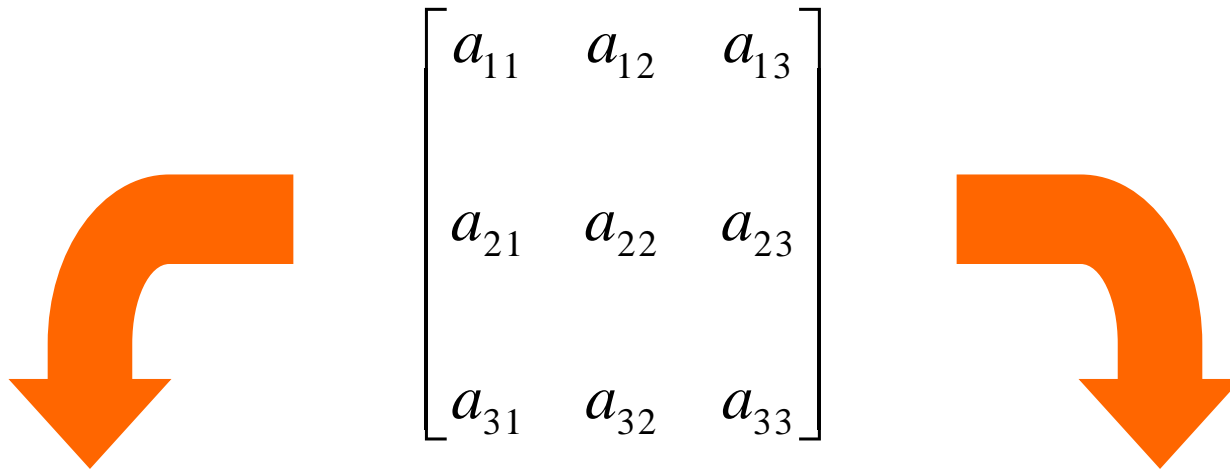
Solve $Ax = b$

Step 1 $A = LU \rightarrow$ **LU factorization**

Step 2 $Ly = b \rightarrow$ Solve by **forward substitution**

Step 3 $Ux = y \rightarrow$ Solve by **backward substitution**

Gaussian Elimination



$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a_{32}^{(1)}}{a_{22}^{(1)}} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}$$

Gaussian Elimination

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a_{32}^{(1)}}{a_{22}^{(1)}} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}$$



$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



LU decomposition - Algorithm

For $k = 1$ to $n-1$

Consider the augmented matrix Ab

$$m(k+1:n, k) = a(k+1:n, k) / a(k, k) \leftarrow \text{Pivot}$$

Multiplier \nearrow

$$a(k+1:n, k+1:n+1) = a(k+1:n, k+1:n+1) +$$

$$- m(k+1:n, k) * a(k, k+1:n+1)$$

Note that:

- Elements of matrix A are **not** zeroed
- Multipliers can be stored in matrix A (instead of zeroing the elements of A)

LU decomposition – Algorithm cost

For $k = 1$ to $n-1$ *Consider the augmented matrix Ab*

$$m(k+1:n, k) = \left. \frac{a(k+1:n, k)}{a(k, k)} \right\} \text{(n-k) divisions}$$

$$a(k+1:n, k+1:n+1) = a(k+1:n, k+1:n+1) + \underbrace{-m(k+1:n, k) * a(k, k+1:n+1)}_{\text{(n-k)(n-k+1) multiplications}}$$

Cost:

- **(n-k)(n-k+2)** multiplication and division operations

LU decomposition – Algorithm cost

For $k = 1$ to $n-1$ *Consider the augmented matrix Ab*

$$m(k+1:n, k) = \left. \begin{array}{l} a(k+1:n, k) \\ \hline a(k, k) \end{array} \right\} \text{(n-k) divisions}$$

$$a(k+1:n, k+1:n+1) = a(k+1:n, k+1:n+1) + \underbrace{-m(k+1:n, k) * a(k, k+1:n+1)}_{\text{(n-k)(n-k+1) multiplications}}$$

Cost:

- **(n-k)(n-k+2)** multiplication and division operations

- **n-1 iterations** $\rightarrow \sum_{k=1}^{n-1} (n-k)(n-k+2) = O(n^3)$

LU decomposition – Algorithm cost

Further we have:

- Multiplication and division operations for solving the triangular systems: $O(n^2)$
- Addition and subtraction operations: $O(n^3)$
- *Note that **addition/subtraction** operations have a **lower cost** with respect to **multiplication/division** operations, that is $O(l)$ vs $O(l^2)$ where l is the length of operands*
- *Anyway usually we consider the number of floating point operations **FLOPs***

Limitation to this approach

Gaussian Elimination method **may fail**:

- **Null pivots**: we have a division by zero during formation of the multipliers if a
- **Small pivots** (Round-off error)

Both can be solved with **partial pivoting**

Limitation to this approach

- If at iteration k $a_{kk}^{(k)} = 0$ we cannot form the multipliers

$$m_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \quad i = k+1:n$$

- A simple way to overcome this problem is the following:

If $a_{kk}^{(k)} = 0$

Find $a_{ik}^{(k)} \neq 0$ with $i > k$

Swap row i with row k

Limitation to this approach

Two Important Theorems

- Partial pivoting (swapping rows) always succeeds if M is non singular
- LU factorization applied to a diagonally dominant matrix will never produce a zero pivot

Limitation to this approach

- In practical computation, **also small pivots can give problems**
- **Small pivots** can lead to **large multipliers m_{ik}**
- If m_{ik} is large then there is a possible loss of significance in the subtraction **$a_{ij}^{(k)} - m_{ik}a_{kj}^{(k)}$** , with low-order digits of $a_{ij}^{(k)}$ being lost
- Losing these digits could correspond to making a relatively large ***change to the original matrix A***

Pivoting and Numerical Stability

- Three different **pivoting strategies** to avoid instability
- All three strategies ensure that the multipliers are nicely bounded: $|m_{ik}| < 1 \quad i = k + 1 : n$

Partial pivoting

At k th stage, the k th and r th rows are interchanged, where

$$|a_{rk}^{(k)}| = \max_{k \leq i \leq n} |a_{ik}^{(k)}|$$

- **Pivot of maximal magnitude over the the pivot column** is selected as pivot

Pivoting and Numerical Stability

Complete pivoting

At k th stage rows k and r and columns k and s are swapped

$$\left| a_{rs}^{(k)} \right| = \max_{k \leq i, j \leq n} \left| a_{ij}^{(k)} \right|$$

- Pivot of maximal magnitude over the whole submatrix

Rook pivoting

At k th stage, rows k and r and columns k and s are swapped

$$\left| a_{rs}^{(k)} \right| = \max_{k \leq i \leq n} \left| a_{is}^{(k)} \right| = \max_{k \leq j \leq n} \left| a_{rj}^{(k)} \right|$$

- Pivot of maximal magnitude in both its column and its row

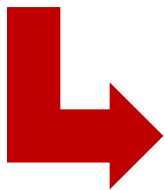
Pivoting and Numerical Stability

Consider the system

$$\begin{bmatrix} 1.25 \cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

After GE

$$\begin{bmatrix} 1.25 \cdot 10^{-4} & 1.25 \\ 0 & 12.5 - (1.25 \cdot 10^5) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 - (6.25 \cdot 10^5) \end{bmatrix}$$



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5\text{digits}} = \begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

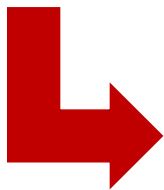
Pivoting and Numerical Stability

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$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5\text{digits}} = \begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{3\text{digits}} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

Pivoting and Numerical Stability

$$\begin{bmatrix} 1.25 \cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

Swap rows

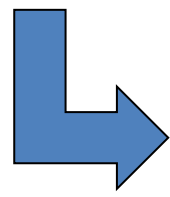
$$\begin{bmatrix} 12.5 & 12.5 \\ 1.25 \cdot 10^{-4} & 1.25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$

GE

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 - 12.5 \cdot 10^{-5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 - 75 \cdot 10^{-5} \end{bmatrix}$$

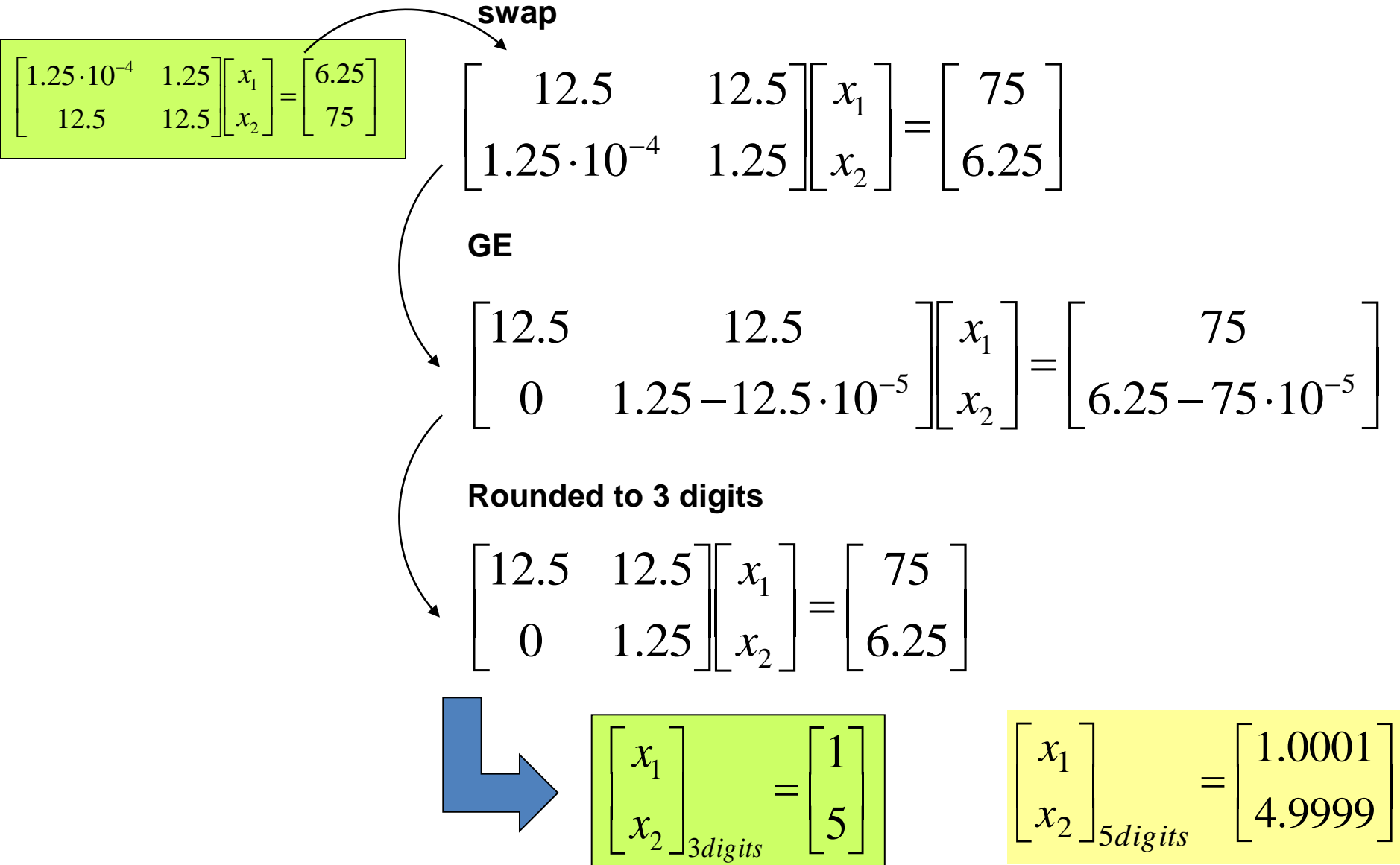
Rounded to 5 digits

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5\text{digits}} = \begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

Pivoting and Numerical Stability



Fill-in Problem


- If we have a **sparse matrix**, Gaussian elimination could **destroy its sparsity**

- Consider the matrix

$$A = \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & 1 & 0 & 0 & 0 \\ \frac{1}{10} & 0 & 1 & 0 & 0 \\ \frac{1}{10} & 0 & 0 & 1 & 0 \\ \frac{1}{10} & 0 & 0 & 0 & 1 \end{bmatrix}$$

Fill-in Problem

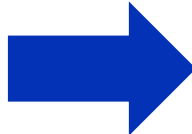
- Example

$$A = \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & 1 & 0 & 0 & 0 \\ \frac{1}{10} & 0 & 1 & 0 & 0 \\ \frac{1}{10} & 0 & 0 & 1 & 0 \\ \frac{1}{10} & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \text{First step of} \\ \text{Gaussian} \\ \text{Elimination} \end{array} \quad \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ 0 & \frac{99}{100} & -\frac{1}{100} & -\frac{1}{100} & -\frac{1}{100} \\ 0 & -\frac{1}{100} & \frac{99}{100} & -\frac{1}{100} & -\frac{1}{100} \\ 0 & -\frac{1}{100} & -\frac{1}{100} & \frac{99}{100} & -\frac{1}{100} \\ 0 & -\frac{1}{100} & -\frac{1}{100} & -\frac{1}{100} & \frac{99}{100} \end{bmatrix}$$


Fill-in Problem

- Example

$$A = \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & 1 & 0 & 0 & 0 \\ \frac{1}{10} & 0 & 1 & 0 & 0 \\ \frac{1}{10} & 0 & 0 & 1 & 0 \\ \frac{1}{10} & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{LU factorization}} \begin{bmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{99}{100} & -\frac{1}{100} & -\frac{1}{100} & -\frac{1}{100} \\ \frac{1}{10} & -\frac{1}{99} & \frac{98}{99} & -\frac{1}{99} & -\frac{1}{99} \\ \frac{1}{10} & -\frac{1}{99} & -\frac{1}{98} & \frac{97}{98} & -\frac{1}{98} \\ \frac{1}{10} & -\frac{1}{99} & -\frac{1}{98} & -\frac{1}{97} & \frac{96}{97} \end{bmatrix}$$

LU factorization


Fill-in Problem

- Reorder the matrix by swapping the first and last row and then the first and last column

$$Pr = Pc = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad PrAPc = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{10} \\ 0 & 1 & 0 & 0 & \frac{1}{10} \\ 0 & 0 & 1 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 1 & \frac{1}{10} \\ 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \end{bmatrix}$$

Fill-in Problem

- After the LU factorization there is not fill-in

$$\begin{array}{c}
 PrAPc = \\
 \left[\begin{array}{ccccc}
 1 & 0 & 0 & 0 & \frac{1}{10} \\
 0 & 1 & 0 & 0 & \frac{1}{10} \\
 0 & 0 & 1 & 0 & \frac{1}{10} \\
 0 & 0 & 0 & 1 & \frac{1}{10} \\
 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10}
 \end{array} \right]
 \end{array}
 \xrightarrow{\text{LU factorization}}
 \begin{array}{c}
 \left[\begin{array}{ccccc}
 1 & 0 & 0 & 0 & \frac{1}{10} \\
 0 & 1 & 0 & 0 & \frac{1}{10} \\
 0 & 0 & 1 & 0 & \frac{1}{10} \\
 0 & 0 & 0 & 1 & \frac{1}{10} \\
 1 & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & -\frac{3}{100}
 \end{array} \right]
 \end{array}$$