# SPARSE MATRICES

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# COMPACT STORAGE FORMAT

Most of the material is from:

L. Formaggia, F. Saleri, A. Veneziani **Solving Numerical PDEs: Problems, Applications, Exercises -** Appendix *The treatment of sparse matrices*

BSR format from:

*https://software.intel.com/en-us/mkl-developer-reference-c-sparse-blas-bsr-matrix-storage-format*

# Storage Methods for Sparse Matrix

- A matrix is **sparse** if it contains a **large number** of zeros
- **sparsity** of the matrix =

  number of zero-valued elements / total number of elements
- **density =** 1 - **sparsity**
- A matrix is **sparse** if its sparsity is > 0.5
- **But** *sparsity is interesting* if in a matrix of size nxn

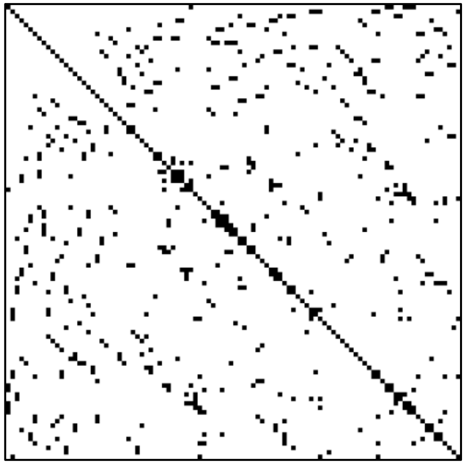
  → the number of non-zero entries is ***O*(*n*)**
- This means that the average number of non-zero entries in each row is bounded independently from *n*
- A non-sparse matrix is said **full** or **dense** if the number of non-zero elements is $O(n^2)$

# Storage Methods for Sparse Matrix

- If the location of the zero elements is known a-priori, we can avoid reserving storage for them

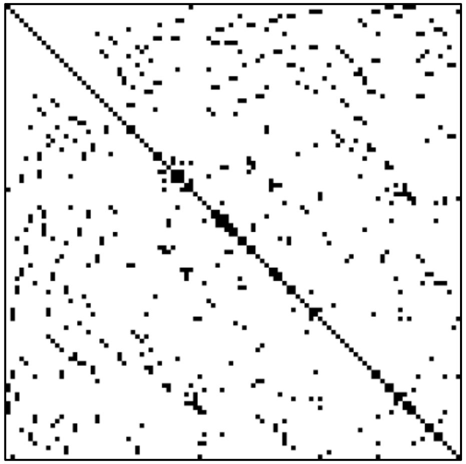
- The distribution of non-zero elements of a sparse matrix may be described by:
  - the **sparsity pattern**, defined as the set $\{(i, j) : A_{ij} = 0\}$
  - the matrix graph, where nodes *i* and *j* are connected by an edge if and only if $A_{ij} = 0$

- In order to take advantage of the large number of zero elements, ***special schemes*** *are required to store sparse matrices*



# Storage Methods for Sparse Matrix

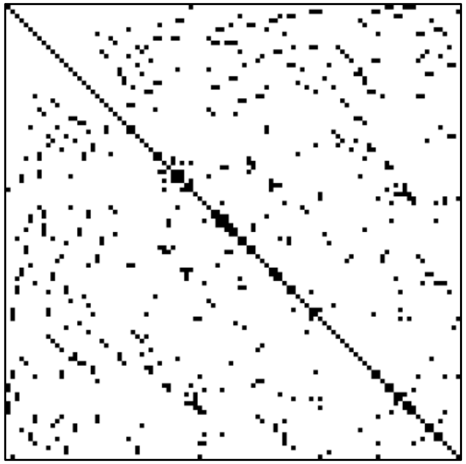
- The use of adequate storage techniques for sparse matrices is fundamental, especially with large-scale problems
- Large sparse matrices often appear in scientific or engineering applications when solving partial differential equations
- Example
  - Suppose we want to solve the Navier-Stokes equations on a two-dimensional grid formed by 10.000 vertexes with finite elements
  - The number of degrees of freedom is around $10^5$ for the pressure and $4\times10^5$ for each component of the velocity
  - The associated matrix will then be 90000×90000
  - If we store all $8.1\times10^9$ coefficients, using double precision (8 bytes), around 60 Gigabytes are necessary!
  - ***This is too much (even for a very large computer)***



# Storage Methods for Sparse Matrix

- In case of a **three-dimensional problem** the situation becomes even **worse**, since the number of degrees of freedom grows very rapidly as the grid gets finer
  - Nowadays it is common to deal with millions of degrees of freedom

- Therefore to store sparse matrices efficiently we need **data formats** that are **more compact** than the classical array



# Storage Methods for Sparse Matrix

- The adoption of sparse formats may affect the ***speed of certain operations***

- For example, with a sparse format we **cannot access** or search for a **particular element** (or group of elements) directly, using the two indexes *i* and *j* to determine where entry $A_{ij}$ is located in the memory
- On the other hand, even if the operation of accessing an entry of a matrix in sparse format turns out to be less efficient, by adopting a sparse format we will nevertheless **access only nonzero elements,** thus **executing only useful operations**

# Storage Methods for Sparse Matrix

- Hence, in general, the sparse format is preferable in terms of storage as well as in terms of computing time, as long as the matrix is **sufficiently sparse**

- The main goal of sparse formats is:
  - to **represent only the nonzero elements**
  - to be able **to efficiently perform the common matrix operations**

# Storage Methods for Sparse Matrix

- We can distinguish different kinds of **operations** on a matrix
- The most important operations are:

1. **accessing** a **generic element** (random access)
2. **accessing** the elements of a **whole row**: important when multiplying a matrix by a vector
3. **accessing** the elements of a **whole column**, or equivalently, of a row in the transpose matrix (relevant for operations such as symmetrizing the matrix after imposing Dirichlet conditions)
4. **adding a new element** to the matrix pattern: this is a critical issue if the pattern is not known beforehand or it can change throughout the computations
5. and the **common operation** of **multiplying a matrix and a vector**

# Storage Methods for Sparse Matrix

- It is important to characterize formats for sparse matrices by the computational cost of these operations and by how the latter depends on the matrix size

- Different formats for sparse matrices exist due to the fact that there is *no format that is simultaneously optimal for all the above operations*, and at the same time *efficient in terms of* ***storage capacity***

# Storage Methods for Sparse Matrix

- In the following:
  - ***n*** is the matrix' size
  - ***nz*** *is* the number of non-zero entries
  - We adopt the convention of indexing entries of matrices and vectors (arrays) starting from 1 (as in Matlab)
  - *Aij* will denote the entry of the matrix A on row *i* and column *j*
- To estimate how much memory the matrix occupies we assume that:
  - an integer occupies 4 bytes
  - a real number (floating point repres.) 8 bytes (double precision)
  - For example, storing a square matrix having $n$ = 12 would require $12 \times 12 \times 8 = 1152$ bytes

# The Coordinate format: COO format

- The simplest storage scheme for sparse matrices is the format by **coordinate**

- The data structure consists of three arrays:
  - **A** - a real array containing all the real (or complex) values of the nonzero elements in **any order**
  - **I** - an integer array containing their row indices
  - **J** - a second integer array containing their column indices
  - I, J and A all have *nz* elements, as many as the number of non-zero elements of the matrix

# The Coordinate format: COO format

**Example**

The matrix A

| | | | | |
|---|---|---|---|---|
| **1.** | 0. | 0. | **2.** | 0. |
| **3.** | **4.** | 0. | **5.** | 0. |
| **6.** | 0. | **7.** | **8.** | **9.** |
| 0. | 0. | **10.** | **11.** | 0. |
| 0. | 0. | 0. | 0. | **12.** |

is represented (*for example*) by

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **A** | **12.** | **9.** | **7.** | **5.** | **1.** | **2.** | **11.** | **3.** | **6.** | **4.** | **8.** | **10.** |
| **I** | **5** | **3** | **3** | **2** | **1** | **1** | **4** | **2** | **3** | **2** | **3** | **4** |
| **J** | **5** | **5** | **3** | **4** | **1** | **4** | **4** | **1** | **1** | **2** | **4** | **3** |

- Notice that elements are listed in an ***arbitrary order***

# The Coordinate format: COO format

**Example** - $n$=12 and $nz$=58

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **101** | **102** | 0 | **103** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **104** | **105** | **106** | **107** | **108** | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | **109** | **110** | 0 | **111** | **112** | 0 | 0 | 0 | 0 | 0 | 0 |
| **113** | **114** | 0 | **115** | **116** | 0 | **117** | 0 | 0 | 0 | 0 | 0 |
| 0 | **118** | **119** | **120** | **121** | **122** | **123** | **124** | 0 | 0 | 0 | 0 |
| 0 | 0 | **125** | 0 | **126** | **127** | 0 | **128** | **129** | 0 | 0 | 0 |
| 0 | 0 | 0 | **130** | **131** | 0 | **132** | **133** | 0 | **134** | 0 | 0 |
| 0 | 0 | 0 | 0 | **135** | **136** | **137** | **138** | **139** | **140** | **141** | 0 |
| 0 | 0 | 0 | 0 | 0 | **142** | 0 | **143** | **144** | 0 | **145** | **146** |
| 0 | 0 | 0 | 0 | 0 | 0 | **147** | **148** | 0 | **149** | **150** | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | **151** | **152** | **153** | **154** | **155** |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **156** | 0 | **157** | **158** |

The space occupied is:

- $8 \times n \times n$ =
  = 8 x 12 x 12 **= 1152** bytes

**in full format**

# The Coordinate format: COO format

**Example -** $n$=12 and $nz$=58

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **101** | **102** | 0 | **103** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **104** | **105** | **106** | **107** | **108** | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | **109** | **110** | 0 | **111** | **112** | 0 | 0 | 0 | 0 | 0 | 0 |
| **113** | **114** | 0 | **115** | **116** | 0 | **117** | 0 | 0 | 0 | 0 | 0 |
| 0 | **118** | **119** | **120** | **121** | **122** | **123** | **124** | 0 | 0 | 0 | 0 |
| 0 | 0 | **125** | 0 | **126** | **127** | 0 | **128** | **129** | 0 | 0 | 0 |
| 0 | 0 | 0 | **130** | **131** | 0 | **132** | **133** | 0 | **134** | 0 | 0 |
| 0 | 0 | 0 | 0 | **135** | **136** | **137** | **138** | **139** | **140** | **141** | 0 |
| 0 | 0 | 0 | 0 | 0 | **142** | 0 | **143** | **144** | 0 | **145** | **146** |
| 0 | 0 | 0 | 0 | 0 | 0 | **147** | **148** | 0 | **149** | **150** | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | **151** | **152** | **153** | **154** | **155** |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **156** | 0 | **157** | **158** |

The space occupied is:

- $8 \times n \times n$ = **1152** bytes **in full format**
- $(4+4+8) \times nz$ = 16 x 58 = **= 928** bytes **in COO format**

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 | . . . |
| | . . . | 121 | 122 | 123 | 124 | 125 | 126 | 127 | 128 | 129 | 130 | 131 | 132 | 133 | 134 | 135 | 136 | 137 | 138 | 139 | . . . |
| | . . . | 140 | 141 | 142 | 143 | 144 | 145 | 146 | 147 | 148 | 149 | 150 | 151 | 152 | 153 | 154 | 155 | 156 | 157 | 158 | |
| I | **1** | **1** | **1** | **2** | **2** | **2** | **2** | **2** | **3** | **3** | **3** | **3** | **4** | **4** | **4** | **4** | **4** | **5** | **5** | **5** | **. . .** |
| | **. . .** | **5** | **5** | **5** | **5** | **6** | **6** | **6** | **6** | **6** | **7** | **7** | **7** | **7** | **7** | **8** | **8** | **8** | **8** | **8** | **. . .** |
| | **. . .** | **8** | **8** | **9** | **9** | **9** | **9** | **9** | **10** | **10** | **10** | **10** | **11** | **11** | **11** | **11** | **11** | **12** | **12** | **12** | |
| J | 1 | 2 | 4 | 1 | 2 | 3 | 4 | 5 | 2 | 3 | 5 | 6 | 1 | 2 | 4 | 5 | 7 | 2 | 3 | 4 | . . . |
| | . . . | 5 | 6 | 7 | 8 | 3 | 5 | 6 | 8 | 9 | 4 | 5 | 7 | 8 | 10 | 5 | 6 | 7 | 8 | 9 | . . . |
| | . . . | 10 | 11 | 6 | 8 | 9 | 11 | 12 | 7 | 8 | 10 | 11 | 8 | 9 | 10 | 11 | 12 | 9 | 11 | 12 | |

# The Coordinate format: COO format

- COO format *does not guarantee rapid access to an element, nor to rows or columns*

- Finding the generic element of the matrix from the row and column indexes normally requires a number of operations proportional to *nz*
- In fact, it is necessary to go through all elements of I and J until one hits those indexes, using expensive comparison operations

- Using specific techniques to store the indexes in special search data structure, it is possible to reduce the cost to $O(\log_2(nz))$, but at a higher storing price

# The Coordinate format: COO format

- The operation of **multiplying** a matrix A and a vector **x** can be done directly, by running through the elements of the three arrays

- A possible code for the product **y** = A**x** using MATLAB

```
y=zeros(nz,1);
for k=1:nz
  i=I(k);
  j=J(k);
  y(i)= y(i) + A(k)*x(j); % notice the use of i and j
end
```

# The Coordinate format: COO format

Observations

- The additional cost of this operation (compared to the analogue for a full matrix) depends essentially on indirect addressing:
  - accessing **`y(i)`** requires first of all to access **`I(k)`**

- The access and update of arrays **`x`** and **`y`** does not proceed by consecutive elements → the possibility of optimizing the use of the processor's cache is greatly reduced

# The Coordinate format: COO format

**Observations**

- Operations are performed only on non-zero elements and in general we have $nz << n^2$

- An advantage of this format is that:
  - It is easy to add a new element to the matrix
  - In fact, it is enough to add a new entry to the arrays I, J and A

- For this reason, COO is often used when the pattern is not known a priori

# The *skyline* format

- The format called **skyline** is among the first used to store matrices arising from the method of finite elements
- The idea is to store the area formed by the **elements between the first and last non-zero coefficient**, on **each row**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 102 | **0** | 103 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 104 | 105 | 106 | 107 | 108 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 109 | 110 | **0** | 111 | 112 | 0 | 0 | 0 | 0 | 0 | 0 |
| 113 | 114 | **0** | 115 | 116 | **0** | 117 | 0 | 0 | 0 | 0 | 0 |
| 0 | 118 | 119 | 120 | 121 | 122 | 123 | 124 | 0 | 0 | 0 | 0 |
| 0 | 0 | 125 | **0** | 126 | 127 | **0** | 128 | 129 | 0 | 0 | 0 |
| 0 | 0 | 0 | 130 | 131 | **0** | 132 | 133 | **0** | 134 | 0 | 0 |
| 0 | 0 | 0 | 0 | 135 | 136 | 137 | 138 | 139 | 140 | 141 | 0 |
| 0 | 0 | 0 | 0 | 0 | 142 | **0** | 143 | 144 | **0** | 145 | 146 |
| 0 | 0 | 0 | 0 | 0 | 0 | 147 | 148 | **0** | 149 | 150 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 151 | 152 | 153 | 154 | 155 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 156 | **0** | 157 | 158 |

- This forces to **store some null entries**

# The *skyline* format

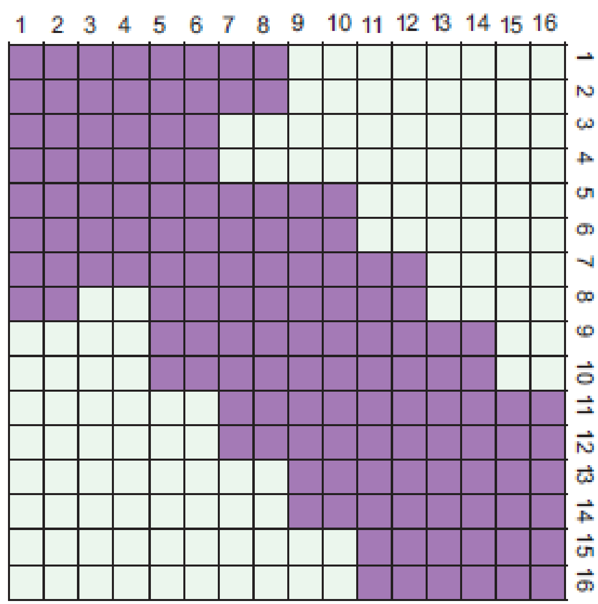
- This extra cost will be small if the matrix has non-zero entries clustered around the diagonal

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 102 | 0 | 103 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 104 | 105 | 106 | 107 | 108 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 109 | 110 | 0 | 111 | 112 | 0 | 0 | 0 | 0 | 0 | 0 |
| 113 | 114 | 0 | 115 | 116 | 0 | 117 | 0 | 0 | 0 | 0 | 0 |
| 0 | 118 | 119 | 120 | 121 | 122 | 123 | 124 | 0 | 0 | 0 | 0 |
| 0 | 0 | 125 | 0 | 126 | 127 | 0 | 128 | 129 | 0 | 0 | 0 |
| 0 | 0 | 0 | 130 | 131 | 0 | 132 | 133 | 0 | 134 | 0 | 0 |
| 0 | 0 | 0 | 0 | 135 | 136 | 137 | 138 | 139 | 140 | 141 | 0 |
| 0 | 0 | 0 | 0 | 0 | 142 | 0 | 143 | 144 | 0 | 145 | 146 |
| 0 | 0 | 0 | 0 | 0 | 0 | 147 | 148 | 0 | 149 | 150 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 151 | 152 | 153 | 154 | 155 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 156 | 0 | 157 | 158 |

- Indeed, algorithms have been developed to cluster non zero elements by permuting the rows and columns of the matrix (see, for example, the Cuthill-McKee algorithm)

# *Skyline* for symmetric matrices

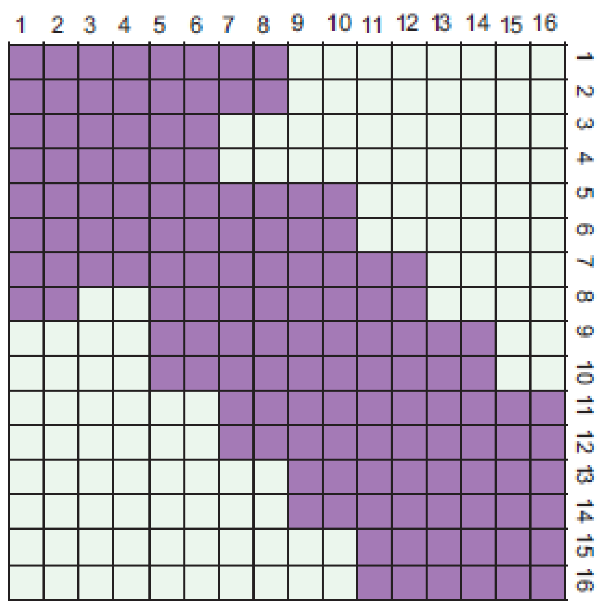
- If a matrix is **symmetric** we can store only:
  - Its **lower triangular part** (diagonal included)
  - **Or** we can store the **diagonal on an auxiliary array** and treat the **off-diagonal entries** separately, having the advantage of allowing the *direct access to the diagonal elements*



# *Skyline* for symmetric matrices

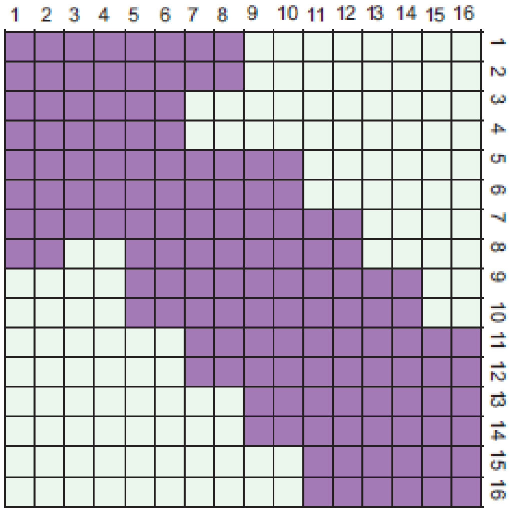
The ***skyline* format** with diagonal array is given by:

- **D** - real array storing diagonal entries
- **AL** - real array storing all *skyline* elements row-wise (except the diagonal)
  This can clearly include null coefficients
- **I** - integer array storing *pointers to rows* of matrix A
  - The $k$th component of array I points to the first element of row $(k + 1)$ in AL



# *Skyline* for symmetric matrices

- All elements of AL from position **I(k)** to **I(k+1)-1** are the off-diagonal elements belonging to row $k + 1$, in column order
- Notice that:
  - ***The first row is not stored***, since it only has the diagonal element
  - **I(k)** points to the first non-zero element on the (k+1)-th row
  - The difference **I(k+1)- I(k)** gives the number of the off-diagonal elements on row k + 1 belonging to the *skyline*



# *Skyline* for symmetric matrices

## Example

- We want to store the symmetric matrix obtained from the lower triangular part of matrix A (seen before) using the *skyline* format
- This matrix can be obtained with the Matlab instruction `tril(A)+ tril(A,-1)'`

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 104 | 0 | 113 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 104 | 105 | 109 | 114 | 118 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 109 | 110 | 0 | 119 | 125 | 0 | 0 | 0 | 0 | 0 | 0 |
| 113 | 114 | 0 | 115 | 120 | 0 | 130 | 0 | 0 | 0 | 0 | 0 |
| 0 | 118 | 119 | 120 | 121 | 126 | 131 | 135 | 0 | 0 | 0 | 0 |
| 0 | 0 | 125 | 0 | 126 | 127 | 0 | 136 | 142 | 0 | 0 | 0 |
| 0 | 0 | 0 | 130 | 131 | 0 | 132 | 137 | 0 | 147 | 0 | 0 |
| 0 | 0 | 0 | 0 | 135 | 136 | 137 | 138 | 143 | 148 | 151 | 0 |
| 0 | 0 | 0 | 0 | 0 | 142 | 0 | 143 | 144 | 0 | 152 | 156 |
| 0 | 0 | 0 | 0 | 0 | 0 | 147 | 148 | 0 | 149 | 153 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 151 | 152 | 153 | 154 | 157 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 156 | 0 | 157 | 158 |

# *Skyline* for symmetric matrices

## Example

- Diagonal **D**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 105 | 110 | 115 | 120 | 121 | 127 | 132 | 138 | 144 | 149 | 154 | 158 |

- Pointer **I** and lower skyline elements **AL**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 104 | 109 | 113 | 114 | 0 | 118 | 119 | 120 | 125 | 0 | 126 | 130 | 131 | 0 | 135 | 136 | 137 | 142 | 0 | 143 | 147 | 148 | 0 | 151 | 152 | 153 | 156 | 0 | 157 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 104 | 0 | 113 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 104 | 105 | 109 | 114 | 118 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 109 | 110 | 0 | 119 | 125 | 0 | 0 | 0 | 0 | 0 | 0 |
| 113 | 114 | 0 | 115 | 120 | 0 | 130 | 0 | 0 | 0 | 0 | 0 |
| 0 | 118 | 119 | 120 | 121 | 126 | 131 | 135 | 0 | 0 | 0 | 0 |
| 0 | 0 | 125 | 0 | 126 | 127 | 0 | 136 | 142 | 0 | 0 | 0 |
| 0 | 0 | 0 | 130 | 131 | 0 | 132 | 137 | 0 | 147 | 0 | 0 |
| 0 | 0 | 0 | 0 | 135 | 136 | 137 | 138 | 143 | 148 | 151 | 0 |
| 0 | 0 | 0 | 0 | 0 | 142 | 0 | 143 | 144 | 0 | 152 | 156 |
| 0 | 0 | 0 | 0 | 0 | 0 | 147 | 148 | 0 | 149 | 153 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 151 | 152 | 153 | 154 | 157 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 156 | 0 | 157 | 158 |

# *Skyline* for symmetric matrices

## Example

- Diagonal **D**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 105 | 110 | 115 | 120 | 121 | 127 | 132 | 138 | 144 | 149 | 154 | 158 |

- Pointers **I** and lower skyline elements **AL**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 104 | 109 | 113 | 114 | 0 | 118 | 119 | 120 | 125 | 0 | 126 | 130 | 131 | 0 | 135 | 136 | 137 | 142 | 0 | 143 | 147 | 148 | 0 | 151 | 152 | 153 | 156 | 0 | 157 |

- Note that in the *n*th place of the array **I** we have left a pointer at the beginning of an hypothetical position. In this way:
  - We can compute the number of skyline elements in the last row, that is `I(n) - I(n-1)`
  - `I(n)- 1` is the **total number of elements** in the skyline

# *Skyline* for symmetric matrices

The product $\mathbf{y} = A\mathbf{x}$ following MATLAB syntax is given by:

```
y=D.*x;
for k=2:n
  nex = I(k)-I(k-1);
  ik = I(k-1):I(k)-1;
  jcol= k-nex:k-1;
  y(k) = y(k)+dot(AL(ik),x(jcol));
  y(jcol)= y(jcol)+AL(ik)*x(k);
end
```

- We ***operate symmetrically on rows and columns*** to exploit the fact that only the lower triangular part was stored in AL

# *Skyline* for symmetric matrices

- The memory needed to store the matrix in this format depends on how effectively the skyline reproduces the actual pattern
- In our case:
  - The **full format** requires: 12 x 12 x 8 = **1152**
  - Array **AL** contains **29 real numbers** (including six 0s)
  - Array **D** of **fixed length *n*=12** containing reals
  - Array **I** of **fixed length *n*=12** containing integers
  - **Total**: (29 + 12) x 8 +n x 4 = **376**

- In general, we need $(n_{AL} + n)$ x 8 + n x 4
- Generally, *Skyline* is more convenient than the COO format if the coefficients are well clustered around the diagonal

# *Skyline* for general matrices

- As for **non-symmetric matrices**, a reasonable way to proceed is to split A into:
  - The diagonal D
  - The strictly lower triangular part E
  - The strictly upper triangular part F

- Using the Matlab syntax, these matrices would be defined as:

```
D=diag(diag(A));
E=tril(A,-1);
F=triu(A,1);
```

# *Skyline* for general matrices

- In general, we need two arrays of indexes: one for pointer to array E and one for pointers to array FT
- If the pattern of A is symmetric, the skyline of E coincides with that of FT, and the same array of pointers I is for both triangular parts
- Diagonal **D**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 105 | 110 | 115 | 121 | 127 | 132 | 138 | 144 | 149 | 154 | 158 |

- Pointer **I**, lower *skyline* elements **E** and upper *skyline* elements **FT**

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 102 | 106 | 0 | 107 | 103 | 116 | 111 | 108 | 122 | 0 | 112 | 0 | 123 | 117 | 133 | 128 | 124 | 139 | 0 | 129 | 140 | 134 | 150 | 145 | 141 | 155 | 156 | 0 | 146 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 104 | 109 | 113 | 114 | 0 | 118 | 119 | 120 | 125 | 0 | 126 | 130 | 131 | 0 | 135 | 136 | 137 | 142 | 0 | 143 | 147 | 148 | 0 | 151 | 152 | 153 | 156 | 0 | 157 |

# *Skyline* for general matrices

- The product matrix-vector **y** $= A$**x** now reads

```
y=D.*x;
for k=2:n
  nex = I(k)-I(k-1);
  ik = I(k-1):I(k)-1;
  jcol = k-nex:k-1;
  y(k) = y(k)+dot(E(ik),x(jcol));
  y(jcol)= y(jcol)+FT(ik)*x(k);
end
```

- `icol` and `ik` contain all indexes corresponding to the columns of row k, so the scalar product `dot(E(ik),x(jcol))` and the multiplication vector-constant `FT(ik)*x(k)` are optimized

# *Skyline* for general matrices

- Notice that in this format the access to diagonal entries is direct
- Being able to access diagonal entries directly has certain advantages:
  - For instance there are methods that, to impose essential boundary condition, only need the access to diagonal elements
- The cost of extracting a row is independent of the matrix' size
- The fact that the data relative to a row are stored consecutively in the memory allows the system to optimize the processor's cache memory when multiplying a matrix by a vector
- The extraction of column is an **expensive** operation that requires many comparisons, and whose cost grows linearly in *n*

# The Compressed Sparse Row (CSR) format

- The problem with the *skyline* format is that the memory used depends on the numeration of elements and is in general impossible to avoid the unnecessary storage of zero elements

- The CSR (Compressed Sparse Row) format can be seen as a compressed version of COO, and also as an improved *skyline*, that stores non-zero elements only

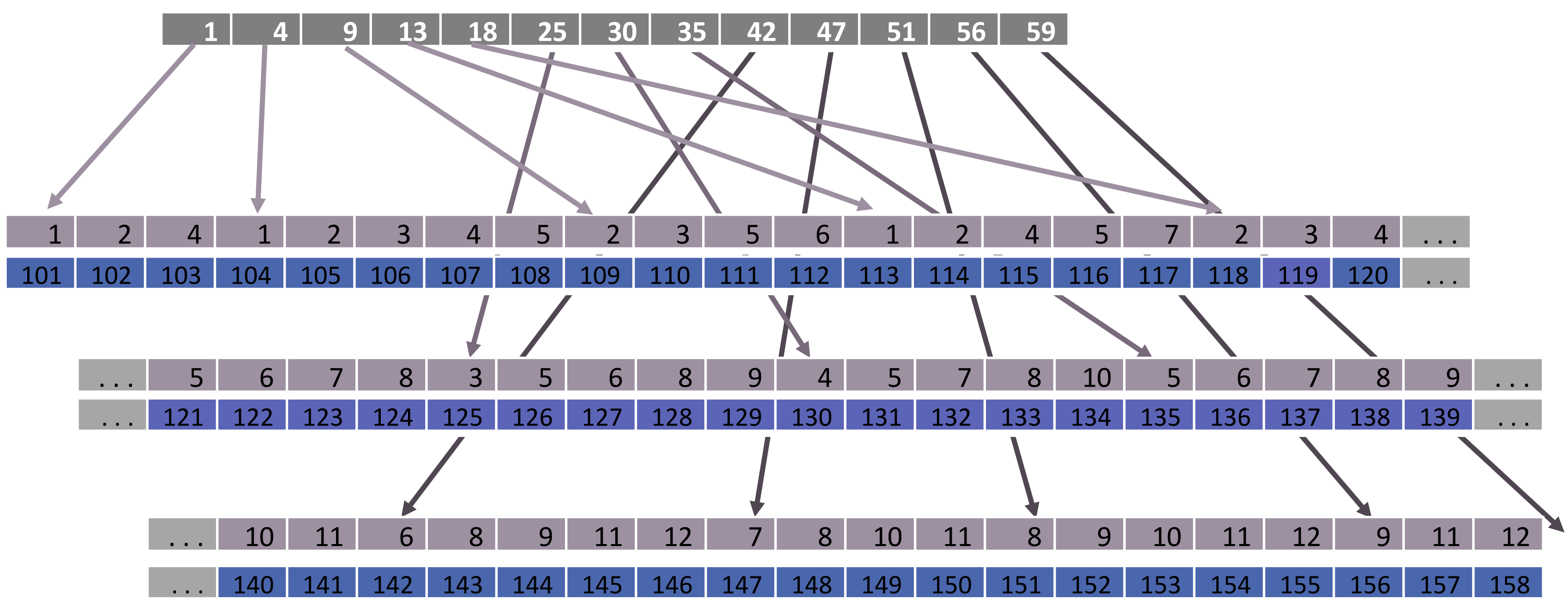
# The Compressed Sparse Row (CSR) format

The CSR format uses three arrays:

- **A** - real array of length *nz* storing the non-zero entries of the matrix, ordered row-wise. It coincides with array A of the COO format
- **J** - integer array of length *nz*, whose entry J(*k*) indicates the column of the element A(*k*). It coincides with array J of the COO format
- **I** - integer array of length *n* containing *pointers* to the rows. Then `I(k)` gives the position where the *k*-th row in A and J begins

# The Compressed Sparse Row (CSR) format

- Array I is usually of length $n + 1$, so that the number of non-zero entries on row $k$ is always `I(k+1)-1-I(k)`
- To make this hold, the last element `I(n+1)` will contain $nz + 1$ and in this way we also have that `nz=I(n+1)-I(1)`



# The Compressed Sparse Row (CSR) format

- The CSR format uses $8 \times nz + 4 \times (nz + n + 1)$ bytes
- CSR format suits square and rectangular matrices alike
- Operations:
  - quick **extraction of row *i*** → elements between **`I(i)`** and **`I(i+1)-1`**
  - **column extraction** requires localizing on *each row* the values of **`J`** corresponding to the wanted column
    - If we adopt no particular ordering, the cost operation is proportional to *nz*
    - If, instead, column indexes of each row in J are ordered in increasing order as in our example, with a binary-search algorithm the extraction cost for a column becomes proportional to $n \log_2(m)$, where $m$ is the mean number of elements on each row
  - the **access to a generic element** has normally a cost proportional to $m$, yet if we order columns it reduces to $\log_2 m$

# The Compressed Sparse Row (CSR) format

The matrix-vector product $\mathbf{y} = A\mathbf{x}$ is given by

```
y=zeros(n,1);
  % y=A(I(1:n)).*x if the diagonal is stored first
for k=1:n
  ik=I(k):I(k+1)-1;
  % ik=I(k)+1:I(k+1)-1; if the diagonal is stored first
  jcol =J(ik); y(k)=y(k)+dot(A(ik),x(jcol));
end
```

# The CSC (Compressed Sparse Column) format

- The **CSC (Compressed Sparse Column) format** stores matrices by ordering them column-wise
- It is easy to extract a column as opposed to rows
- The roles of vectors I and J is exchanged compared with the CSR format
- When performing matrix-vector multiplication with a sparse matrix in CSC format it is preferable to compute the result as a linear combination of the columns of the matrix
- Indeed, if $\mathbf{c}_i$ indicates the $i$-th column of matrix A, we have that $A\mathbf{x} = \Sigma_i\, x_i\mathbf{c}_i$

# The CSC (Compressed Sparse Column) format

- Therefore, the matrix-vector product $\mathbf{y} = A\mathbf{x}$ on a CSC matrix may be computed as:

```
y=zeros(n,1);
for k=1:n
  xcoeff=x(k);
  jk=I(k):I(k+1)-1;
  ik=J(jk);
  y(ik)=y(ik) + xcoeff * A(jk)';
end
```

# The MSR (Modified Sparse Row) format

- The **MSR (Modified Sparse Row)** format is a special version of CSR for square matrices exploiting the fact that:
  - The diagonal elements of many matrices are usually nonzero (matrices generated by finite elements)
  - The diagonal elements are accessed more often than the rest of the elements
- **Diagonal** entries can be stored in one single array, since their *indexes are implicitly known* from their position in the array
- As for the symmetric skyline, only off-diagonal elements are stored in a special fashion, i.e. through a format akin to CSR

# The MSR (Modified Sparse Row) format

The MSR format uses two arrays:

- **V** - real array of values:
  - In the first *n* entries of V we store the diagonal
  - The place *n*+1 in V is left with ***no significant value*** (may sometimes carry some information concerning the matrix)
  - From place *n*+2 onwards off-diagonal elements are stored, row-wise
  - V has length $nz + 1$
- **B** - Bind
  - B has the same length as V → $nz + 1$
  - The first $n + 1$ point to where rows begin
  - From $n$+2 to $nz$+1 there are the column indexes of the elements stored in the corresponding places in V

# The MSR (Modified Sparse Row) format

## Example

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 102 | 0 | 103 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 104 | 105 | 106 | 107 | 108 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 109 | 110 | 0 | 111 | 112 | 0 | 0 | 0 | 0 | 0 | 0 |
| 113 | 114 | 0 | 115 | 116 | 0 | 117 | 0 | 0 | 0 | 0 | 0 |
| 0 | 118 | 119 | 120 | 121 | 122 | 123 | 124 | 0 | 0 | 0 | 0 |
| 0 | 0 | 125 | 0 | 126 | 127 | 0 | 128 | 129 | 0 | 0 | 0 |
| 0 | 0 | 0 | 130 | 131 | 0 | 132 | 133 | 0 | 134 | 0 | 0 |
| 0 | 0 | 0 | 0 | 135 | 136 | 137 | 138 | 139 | 140 | 141 | 0 |
| 0 | 0 | 0 | 0 | 0 | 142 | 0 | 143 | 144 | 0 | 145 | 146 |
| 0 | 0 | 0 | 0 | 0 | 0 | 147 | 148 | 0 | 149 | 150 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 151 | 152 | 153 | 154 | 155 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 156 | 0 | 157 | 158 |

- **Array B**

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 14 | 16 | 20 | 23 | 27 | 33 | 37 | 41 | 51 | 54 | 58 | 60 | 2 | 4 | 1 | 3 | 4 | 5 | 2 | 5 | ... |

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ... | 6 | 1 | 2 | 5 | 7 | 9 | 2 | 3 | 4 | 6 | 7 | 8 | 3 | 5 | 8 | 9 | 4 | 5 | 8 | ... |

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ... | 10 | 5 | 6 | 7 | 9 | 10 | 11 | 6 | 8 | 11 | 12 | 7 | 8 | 11 | 8 | 9 | 10 | 12 | 9 | 11 |

- **Array V**

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 105 | 110 | 115 | 121 | 127 | 132 | 138 | 144 | 149 | 154 | 158 | * | 102 | 103 | 104 | 106 | 107 | 108 | 109 | ... |

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ... | 111 | 112 | 113 | 114 | 116 | 117 | 118 | 119 | 120 | 122 | 123 | 124 | 125 | 126 | 128 | 129 | 130 | 131 | 133 | ... |

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ... | 134 | 135 | 136 | 137 | 139 | 140 | 141 | 142 | 143 | 145 | 146 | 147 | 148 | 150 | 151 | 152 | 153 | 155 | 156 | 157 |

# The MSR (Modified Sparse Row) format

- The MSR format turns out to be **very efficient** in memory terms
- It is one of the most ***compact*** formats for sparse matrices
- It is used in several linear algebra libraries for large problems
- The *drawback* is that it ***only*** applies to *square matrices*
- The matrix-vector product is coded as:

```
y=V(1:n).*x;
for k=1:n
  ik=B(k):B(k+1)-1;
  jcol =B(ik);
  y(k)=y(k)+dot(A(ik),x(jcol));
end
```

# BSR (Block Sparse Row) format

- The **BSR format** is a CSR with dense submatrices of fixed shape instead of scalar items
- The block size must evenly divide the shape of the matrix

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 102 | 0 | 103 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 104 | 105 | 106 | 107 | 108 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 109 | 110 | 0 | 111 | 112 | 0 | 0 | 0 | 0 | 0 | 0 |
| 113 | 114 | 0 | 115 | 116 | 0 | 117 | 0 | 0 | 0 | 0 | 0 |
| 0 | 118 | 119 | 120 | 121 | 122 | 123 | 124 | 0 | 0 | 0 | 0 |
| 0 | 0 | 125 | 0 | 126 | 127 | 0 | 128 | 129 | 0 | 0 | 0 |
| 0 | 0 | 0 | 130 | 131 | 0 | 132 | 133 | 0 | 134 | 0 | 0 |
| 0 | 0 | 0 | 0 | 135 | 136 | 137 | 138 | 139 | 140 | 141 | 0 |
| 0 | 0 | 0 | 0 | 0 | 142 | 0 | 143 | 144 | 0 | 145 | 146 |
| 0 | 0 | 0 | 0 | 0 | 0 | 147 | 148 | 0 | 149 | 150 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 151 | 152 | 153 | 154 | 155 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 156 | 0 | 157 | 158 |

| | | | |
|---|---|---|---|
| A | B | | |
| C | D | E | |
| | F | G | H |
| | | I | L |

- In this example the block size is 3 x 3

# BSR (Block Sparse Row) format

- The BSR format store the **non-zero blocks** of the sparse matrix
- A **non-zero block** is the block that contains *at least one non-zero element*

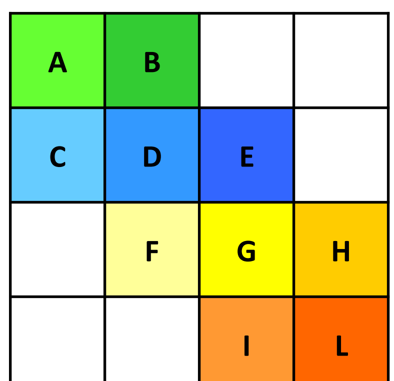
| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 102 | 0 | 103 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 104 | 105 | 106 | 107 | 108 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 109 | 110 | 0 | 111 | 112 | 0 | 0 | 0 | 0 | 0 | 0 |
| 113 | 114 | 0 | 115 | 116 | 0 | 117 | 0 | 0 | 0 | 0 | 0 |
| 0 | 118 | 119 | 120 | 121 | 122 | 123 | 124 | 0 | 0 | 0 | 0 |
| 0 | 0 | 125 | 0 | 126 | 127 | 0 | 128 | 129 | 0 | 0 | 0 |
| 0 | 0 | 0 | 130 | 131 | 0 | 132 | 133 | 0 | 134 | 0 | 0 |
| 0 | 0 | 0 | 0 | 135 | 136 | 137 | 138 | 139 | 140 | 141 | 0 |
| 0 | 0 | 0 | 0 | 0 | 142 | 0 | 143 | 144 | 0 | 145 | 146 |
| 0 | 0 | 0 | 0 | 0 | 0 | 147 | 148 | 0 | 149 | 150 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 151 | 152 | 153 | 154 | 155 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 156 | 0 | 157 | 158 |

| | | | |
|---|---|---|---|
| A | B | | |
| C | D | E | |
| | F | G | H |
| | | I | L |

# BSR (Block Sparse Row) format

The **BSR format** consists of four arrays:

- **Values** - real array containing the elements of the non-zero blocks of a sparse matrix
  - The elements are stored block-by-block in row-major order
  - All elements of non-zero blocks are stored, even if some of them are equal to zero
  - Within each non-zero block elements are stored in column-major order in the case of one-based indexing, and in row-major order in the case of the zero-based indexing



# BSR (Block Sparse Row) format

The **BSR format** consists of four arrays:

- **Columns** - integer array where element *i* is the number of the column in the block matrix that contains the *i*-th non-zero block
- **PointerB** - integer array where element *j* gives the index of the element in the *columns* array that is first non-zero block in row *j* of the block matrix
- **PointerE** - integer array where element *j* gives the index of the element in the *columns* array that contains the last non-zero block in a row *j* of the block matrix plus 1

# BSR (Block Sparse Row) format

## Example

- Values

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 104 | 0 | 102 | 105 | 109 | 0 | 106 | 110 | 103 | 107 | 0 | 0 | 108 | 111 | 0 | 0 | 112 | . . . |

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| . . . | 113 | 0 | 0 | 114 | 118 | 0 | 0 | 119 | 125 | 115 | 120 | 0 | 116 | 121 | 126 | 0 | 122 | 127 | . . . |

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| . . . | 117 | 123 | 0 | 0 | 124 | 128 | 0 | 0 | 129 | 130 | 0 | 0 | 131 | 135 | 0 | 0 | 136 | 142 | . . . |

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| . . . | 132 | 137 | 0 | 133 | 138 | 143 | 0 | 139 | 144 | 134 | 140 | 0 | 0 | 141 | 145 | 0 | 0 | 146 | . . . |

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| . . . | 147 | 0 | 0 | 148 | 151 | 0 | 0 | 152 | 156 | 149 | 153 | 0 | 150 | 154 | 157 | 0 | 155 | 158 |

- Columns

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 2 | 3 | 2 | 3 | 4 | 3 | 4 |

- PointerB

| | | | |
|---|---|---|---|
| 1 | 3 | 6 | 9 |

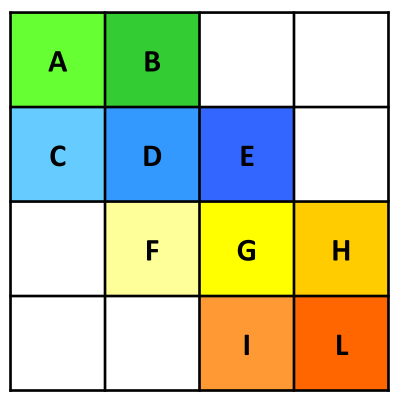
- PointerE

| | | | |
|---|---|---|---|
| 3 | 6 | 9 | 11 |

| | | | |
|---|---|---|---|
| A | B | | |
| C | D | E | |
| | F | G | H |
| | | I | L |

# BSR (Block Sparse Row) format

- The length of the ***values*** array is equal to the number of all elements in the non-zero blocks
- The length of the ***columns*** array is equal to the number of non-zero blocks
- The length of the ***pointerB*** and ***pointerE*** arrays is equal to the number of block rows in the block matrix



# Diagonal format

- Diagonally structured matrices are matrices whose nonzero elements are located along a small number of diagonals

The **diag format** consist of:

- **DIAG** - a rectangular real array storing the diagonals
  - DIAG has size *n* x *Nd*, where *Nd* is the number of diagonals
- **IOFF** - an integer array containing the offsets of each diagonal with respect to the main diagonal
  - IOFF ha size *Nd*

# Diagonal format

- The **order** in which the diagonals are stored in of DIAG is generally **unimportant**
- Since several more **operations** are performed with the **main diagonal**, storing it in the **first column** may be slightly advantageous

- Note that all the diagonals except the main diagonal have fewer than n elements, so there are positions in DIAG that will not be used

# Diagonal format

**Example**

- Matrix

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 102 | 0 | 103 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 104 | 105 | 106 | 107 | 108 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 109 | 110 | 0 | 111 | 112 | 0 | 0 | 0 | 0 | 0 | 0 |
| 113 | 114 | 0 | 115 | 116 | 0 | 117 | 0 | 0 | 0 | 0 | 0 |
| 0 | 118 | 119 | 120 | 121 | 122 | 123 | 124 | 0 | 0 | 0 | 0 |
| 0 | 0 | 125 | 0 | 126 | 127 | 0 | 128 | 129 | 0 | 0 | 0 |
| 0 | 0 | 0 | 130 | 131 | 0 | 132 | 133 | 0 | 134 | 0 | 0 |
| 0 | 0 | 0 | 0 | 135 | 136 | 137 | 138 | 139 | 140 | 141 | 0 |
| 0 | 0 | 0 | 0 | 0 | 142 | 0 | 143 | 144 | 0 | 145 | 146 |
| 0 | 0 | 0 | 0 | 0 | 0 | 147 | 148 | 0 | 149 | 150 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 151 | 152 | 153 | 154 | 155 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 156 | 0 | 157 | 158 |

Element DIAG(i, j) is located in position $a_{i,i+ioff(j)}$ of the original matrix

- DIAG

| | | | | | | |
|---|---|---|---|---|---|---|
| 113 | 0 | 104 | 101 | 102 | 0 | 103 |
| 118 | 114 | 109 | 105 | 106 | 107 | 108 |
| 125 | 119 | 0 | 110 | 0 | 111 | 112 |
| 130 | 0 | 120 | 115 | 116 | 0 | 117 |
| 135 | 131 | 126 | 121 | 122 | 123 | 124 |
| 142 | 136 | 0 | 127 | 0 | 128 | 129 |
| 147 | 0 | 137 | 132 | 133 | 0 | 134 |
| 151 | 148 | 143 | 138 | 139 | 140 | 141 |
| 156 | 152 | 0 | 144 | 0 | 145 | 146 |
| 0 | 0 | 153 | 149 | 150 | 0 | 0 |
| 0 | 0 | 157 | 154 | 155 | 0 | 0 |
| 0 | 0 | 0 | 158 | 0 | 0 | 0 |

IOFF

| | | | | | | |
|---|---|---|---|---|---|---|
| -3 | -2 | -1 | 0 | 1 | 2 | 3 |

# Ellpack-Itpack format

- The **Ellpack-Itpack format** is a general scheme, popular on vector machines

- The Ellpack-Itpack format consists of two rectangular arrays:
  - **COEF** - real array (similar to DIAG) that contains the nonzero elements of A (completing the row by zeros as necessary)
  - **JCOEF** - integer array that contains the column positions of each entry in COEF
  - COEF and JCOEF have size ***n x Nd***, where *n* is the number of rows and *Nd* is the maximum number of nonzero elements per row, whith *Nd* small

# Ellpack-Itpack format

## Example

- Matrix

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 101 | 102 | 0 | 103 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 104 | 105 | 106 | 107 | 108 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 109 | 110 | 0 | 111 | 112 | 0 | 0 | 0 | 0 | 0 | 0 |
| 113 | 114 | 0 | 115 | 116 | 0 | 117 | 0 | 0 | 0 | 0 | 0 |
| 0 | 118 | 119 | 120 | 121 | 122 | 123 | 124 | 0 | 0 | 0 | 0 |
| 0 | 0 | 125 | 0 | 126 | 127 | 0 | 128 | 129 | 0 | 0 | 0 |
| 0 | 0 | 0 | 130 | 131 | 0 | 132 | 133 | 0 | 134 | 0 | 0 |
| 0 | 0 | 0 | 0 | 135 | 136 | 137 | 138 | 139 | 140 | 141 | 0 |
| 0 | 0 | 0 | 0 | 0 | 142 | 0 | 143 | 144 | 0 | 145 | 146 |
| 0 | 0 | 0 | 0 | 0 | 0 | 147 | 148 | 0 | 149 | 150 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 151 | 152 | 153 | 154 | 155 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 156 | 0 | 157 | 158 |

COEF

| | | | | | | |
|---|---|---|---|---|---|---|
| 101 | 102 | 103 | 0 | 0 | 0 | 0 |
| 104 | 105 | 106 | 107 | 108 | 0 | 0 |
| 109 | 110 | 111 | 112 | 0 | 0 | 0 |
| 113 | 114 | 115 | 116 | 117 | 0 | |
| 118 | 119 | 120 | 121 | 122 | 123 | 124 |
| 125 | 126 | 127 | 128 | 129 | 0 | 0 |
| 130 | 131 | 132 | 133 | 134 | 0 | 0 |
| 135 | 136 | 137 | 138 | 139 | 140 | 141 |
| 142 | 143 | 144 | 145 | 146 | 0 | 0 |
| 147 | 148 | 149 | 150 | 0 | 0 | 0 |
| 151 | 152 | 153 | 154 | 155 | 0 | 0 |
| 156 | 157 | 158 | 0 | 0 | 0 | 0 |

JCOEF

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 0 | 0 | 0 | 0 |
| 1 | 2 | 3 | 4 | 5 | 0 | 0 |
| 2 | 3 | 5 | 6 | 0 | 0 | 0 |
| 1 | 2 | 4 | 5 | 7 | 0 | |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 5 | 6 | 8 | 9 | 0 | 0 |
| 4 | 5 | 7 | 8 | 10 | 0 | 0 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 8 | 9 | 11 | 12 | 0 | 0 |
| 7 | 8 | 10 | 11 | 0 | 0 | 0 |
| 8 | 9 | 10 | 11 | 12 | 0 | 0 |
| 9 | 11 | 12 | 0 | 0 | 0 | 0 |

# MATLAB AND SPARSE MATRICES

Material from:

https://it.mathworks.com/help/matlab/math/constructing-sparse-matrices.html

# Matlab and Sparse Matrices

- MATLAB *never creates sparse matrices automatically*
- A representation of the pattern is given by the command **`spy`**

- You must determine if a matrix contains a large enough percentage of zeros to benefit from sparse techniques

- The **density** of a matrix is the ***number of nonzero elements divided by the total number of matrix elements***
- For matrix M, this would be

  **`nnz(M)/prod(size(M))`** or **`nnz(M) / numel(M)`**
- Matrices with very low density are often good candidates for use of the sparse format

# Matlab and Sparse Matrices

## Converting Full to Sparse

- You can convert a full matrix to sparse storage using the **sparse** function with a single argument

```
S = sparse(A)
```

- For example, given the matrix A:

```
A = [ 0 0 0 5
      0 2 0 0
      1 3 0 0
      0 0 4 0];
```

```
S = sparse(A)
```

produces:

```
(3,1)    1
(2,2)    2
(3,2)    3
(4,3)    4
(1,4)    5
```

- Output: nonzero elements of S, with their row and column indices
- The elements are sorted by columns

# Matlab and Sparse Matrices

## Converting Full to Sparse

- You can convert a sparse matrix to full storage using the `full` function, provided the matrix order is not too large
- For example `A = full(S)` reverses the example conversion

- Converting a full matrix to sparse storage is **not** the most frequent way of generating sparse matrices

- If the order of a matrix is small enough that full storage is possible, then *conversion to sparse storage rarely offers significant savings*

# Matlab and Sparse Matrices

## Creating Sparse Matrices Directly

- You can create a sparse matrix from a list of nonzero elements using the **`sparse`** function with five arguments

```
S = sparse(i,j,s,m,n)
```

  where
  - **`i`** and **`j`** are vectors of row and column indices, respectively, for the nonzero elements of the matrix
  - **`s`** is a vector of nonzero values whose indices are specified by the corresponding (i,j) pairs
  - **`m`** is the row dimension for the resulting matrix
  - **`n`** is the column dimension
- The matrix S of the previous example can be generated with:

```
S = sparse([3 2 3 4 1],[1 2 2 3 4],[1 2 3 4 5],4,4)
```

# Matlab and Sparse Matrices

## Creating Sparse Matrices Directly

- The matrix representation of the second difference operator is a tridiagonal matrix with -2s on the diagonal and 1s on the super- and sub-diagonal

- One way to generate it is:

```
D = sparse(1:n,1:n,-2*ones(1,n),n,n);
E = sparse(2:n,1:n-1,ones(1,n-1),n,n);
S = E+D+E'
```

# Matlab and Sparse Matrices

## Creating Sparse Matrices Directly

For n = 5, MATLAB responds with

```
S =
    (1,1)      -2
    (2,1)       1
    (1,2)       1
    (2,2)      -2
    (3,2)       1
    (2,3)       1
    (3,3)      -2
    (4,3)       1
    (3,4)       1
    (4,4)      -2
    (5,4)       1
    (4,5)       1
    (5,5)      -2
```

The full command

**F = full(S)**

displays the corresponding full matrix

```
F = full(S)

F =
    -2   1   0   0   0
     1  -2   1   0   0
     0   1  -2   1   0
     0   0   1  -2   1
     0   0   0   1  -2
```