HOW TO ASSIGN CHANNELS TO STATIONS IN A GRID NETWORK

EFFICIENT USE OF RADIO SPECTRUM IN WIRELESS NETWORKS WITH CHANNEL SEPARATION BETWEEN CLOSE STATIONS

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Course of Networks Algorithms

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- Introduction
- Preliminary Concepts
- Assign Channel to Grids efficiently
  - Hexagonals
  - Bidimensionals
  - Cellulars
- Conclusions

INTRODUCTION

THE PROBLEM

- *Co-Channel reuse distance σ .*
- Minimum distance between stations
- The goal of assigment’s algorithms is to assign channels to stations in a way such that the *Co-Channel Reuse distance* constraint and the *minimum distance beetween close stations* constraint are respected. The number of the channels used must be as small as possible.

THE MODEL

- Graph $G(V,E)$ such that
  V = The stations set
  E = Couples of close stations
- $d(u,v)$ = Distance between vertex u and vertex v
- C = Set of non negative integers
- σ_i = Minimum distance between channels assigned to vertices at distance i

THE MODEL

- $L(\sigma_1, \sigma_2, \ldots, \sigma_{\sigma-1})$*-coloration* of the graph $G(V,E)$ is a function $f : V \to C$ such that:

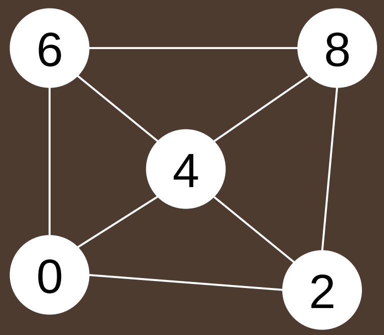
  $|\, f(u) - f(v) \,| \geq \sigma_i$ iff $d(u,v) = i$
- k-$L(\sigma_1, \sigma_2, \ldots, \sigma_{\sigma-1})$*-coloration* of the graph $G(V,E)$ is a function:
  $f : V \to \{1,2, \ldots, k\}$
- $\lambda(G)$ = The biggest color used in an optimal coloration of the graph G
- $\lambda(G)+1$ = The number of colors used

PROBLEMS STUDIED

- We study problems with $\sigma = 3$ and $\sigma = 4$
- *In particular* $L(2,1)$ and $L(2,1,1)$
- Assignment costs

PRELIMINARY CONCEPTS

CLIQUE K_n



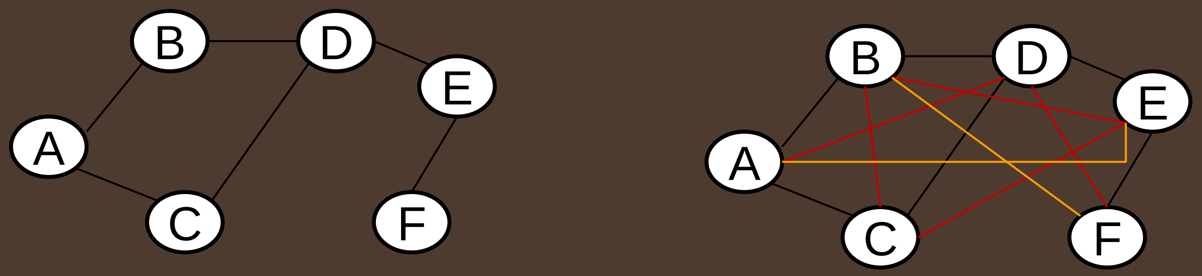
- If the graph G is a clique K_n of n nodes, since the nodes are all adjacent to each other, we have that $\lambda(G) = 2(n - 1)$ for both problems $L(2,1)$ and $L(2,1,1)$
- For the classical vertex coloring problem n colors are requested to color the K_n clique

REDUCTION TO THE CLASSICAL VERTEX COLORING PROBLEM

- Suppose we want to calculate the *L(1,1,...,1)-coloration* of the graph $G(V,E)$.
- We can build the augmented graph $G_\sigma(V, E_\sigma)$, where $E_\sigma = \{ (u,v) \text{ such that } d(u,v) \leq \sigma - 1 \}$

REDUCTION TO THE CLASSICAL VERTEX COLORING PROBLEM

- G　　　　　　　　G_4



- The numbers of colors used, in a classical vertex coloring of the graph G_{σ}, is a lower bound for the numbers of channels used in a *L(1,1,...,1)-coloration* of the graph *G*

REDUCTION TO THE CLASSICAL VERTEX COLORING PROBLEM

- If the graph G_σ have a K_n clique, then n is a lower bound for the number of colors used
- So n is a lower bound also for the number of colors used in a *L(1,1,...1)-coloration* of G
- To refine this bound we need to find the maximum clique of the graph G_σ

LOWER BOUNDS FOR L(k,1,…1)

- A lower bound for the $L(1,1,\ldots 1)$*-coloring* of G is a lower bound for the $L(k,1,\ldots,1)$*-coloring* of G, with $k \geq 1$
- So lower bounds for $L(1,1,1)$ are lower bounds for $L(2,1,1)$ too and lower bounds for $L(1,1)$ are also lower bounds for $L(2,1)$

LEMMA 1

- Consider the *$L(k,1,\ldots,1)$-coloration* of an augmented graph $G(V,E)$, with $k \geq 2$. $\lambda(G) = |V| + 1$ iff G' has an hamiltonian path.
- $G'(V,E')$ is the complementh graph of G, where $E' = \{\ (u,v)$ *such that* (u,v) *do not belongs to* $E\ \}$

PROOF OF LEMMA 1 (FIRST IMPLICATION →)

- If we want to satisfy *the channel separation* constraint, two vertices of G can have consecutive colors iff they are not adjacent. So they are adjacent in G'.
- If $\lambda(G) = |V| + 1$ then there is an ordering $(v_0, v_1, \ldots v_{|V|-1})$ of the vertices such that $f(v_i) = i$
- For what we've seen before, every couple (v_{i-1}, v_i), in that ordered set, is an edge of E'
- So the ordered set $(v_0, v_1, \ldots v_{|V|-1})$ represent an hamiltonian path in G'

PROOF OF LEMMA 1 (SECOND IMPLICATION $\leftarrow$)

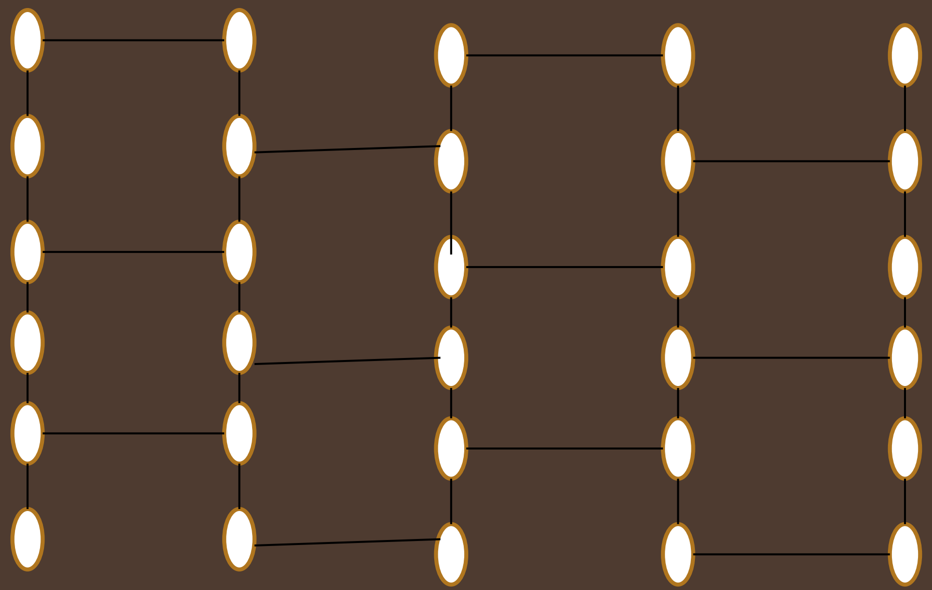
- If G' has an hamiltonian path $(v_0, v_1, \ldots v_{|V|-1})$ then we can build a function $f : V \to \{ 0,1, \ldots |V| -1 \}$ such that $f(v_i) = i$ for every $0 \leq i \leq |V| -1$
- This function is clearly optimal for the *L(k,1,…,1)-coloration* problem of G

LEMMA 2

- Let S_k be a star graph with degree k.
- Let c be the vertex with degree k of the star (the center of the star).
- The biggest color used for the *L(2,1)-coloration* of S is :
  $k + 1 \quad \text{if} \quad f(c) = 0 \text{ or } f(c) = k+1$
  $k + 2 \quad \text{if} \quad 1 \leq f(c) \leq k$

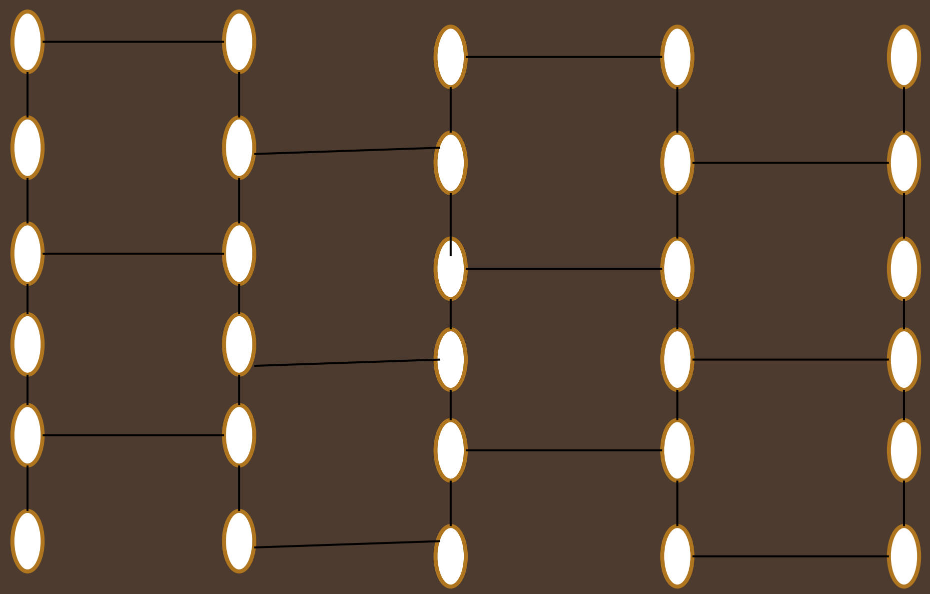
Assign Channels to Grids Efficiently

HEXAGONAL GRIDS



- An hexagonal grid $H(r \cdot c, E)$ is a graph with r rows (from 0 to $r - 1$) and c columns (from 0 to $c - 1$), with $r \geq 3$ and $c \geq 2$.
- A generic vertex u is denoted $u = (i, j)$ where i is his row and j is his column.

HEXAGONAL GRIDS



- Each vertex has degree 3, except for some vertices on the boards.

EDGES OF AN HEXAGONAL GRID

- A vertex (i, j) , which does not belongs to the board of the graph, is adjacent to the following 3 vertices:
- 1 – Vertex $(i - 1, j)$
- 2 – Vertex $(i + 1, j)$
- 3 – Vertex $(i, j + 1)$ or Vertex $(i, j - 1)$ (it depends on whether i and j are both even or odd or one is even and the other is odd)

LEMMA 3

- For $r \geq 3$ and $c \geq 3$ there is a *L(2, 1)-coloration* of an hexagonal grid H of size $r \cdot c$ only if $\lambda(H)=5$
- The proof follows the *Lemma 2*, since there is at least one vertex with degree *3*, that cannot be colored either *0* or *4*.

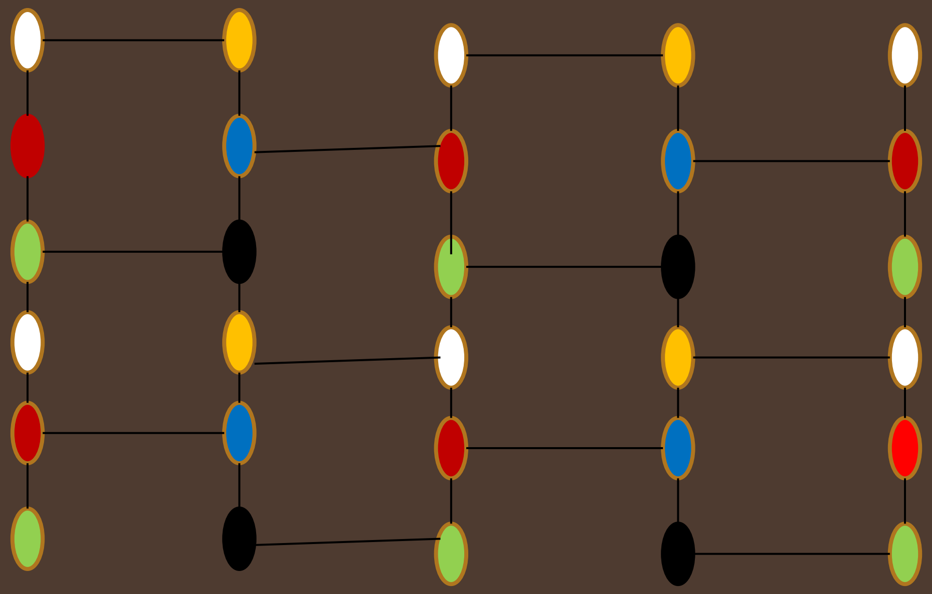
ALGORITHM HEXAGONAL 5-L(2,1) COLORING

IF (($r \geq 3$) AND ($c \geq 3$))

Assign to each vertex $u = (i, j)$ the color $f(u) = (2 \cdot i + 3 \cdot j)$ MOD 6

This algorithm is optimal for hexagonal grids with $r \geq 3$ and $c \geq 3$

ALGORITHM HEXAGONAL 5-L(2,1) COLORING



- White = 0, Black = 1
- Red = 2, Yellow = 3
- Green = 4, Blu = 5

LEMMA 4

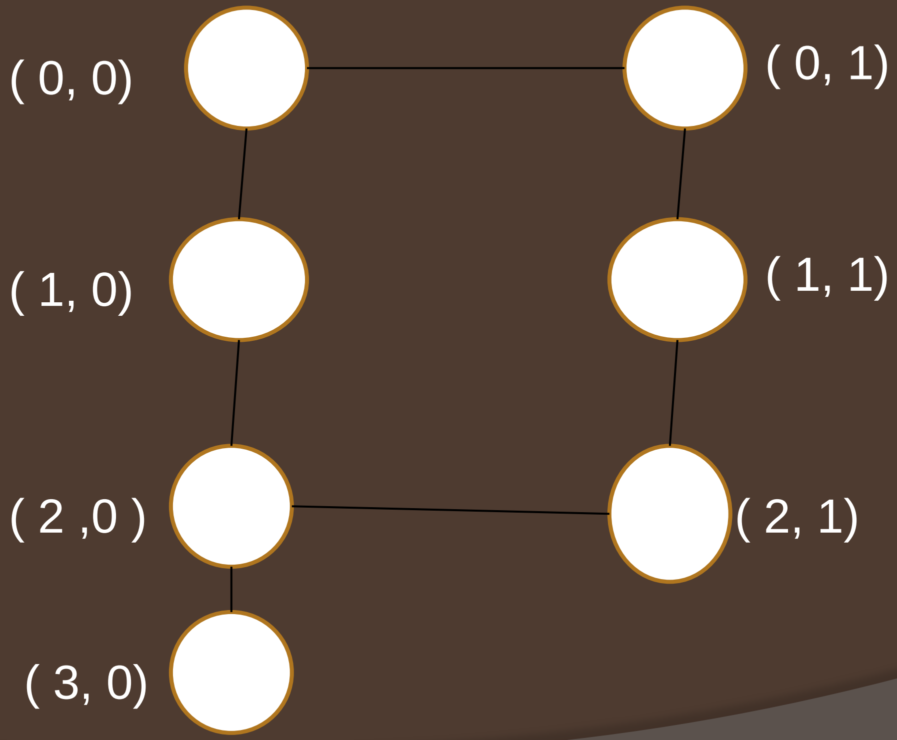
- For $r \geq 3$ and $c \geq 3$, or $r \geq 5$ and $c = 2$, there is a $L(2, 1, 1)$-*coloration* of an hexagonal grid H of size $r \cdot c$ only if $\lambda(H) \geq 6$

PROOF OF LEMMA 4
(CASE r $\geq$ 3 AND c $\geq$ 3)

- Consider the augmented graph $G_4(V,E')$ and his subset:
  $S = \{ (0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1) \}$
- Vertices in the subset S are mutually at distance 3 in H, so they form a clique in G_4
- Therefore, $\lambda(H) > 5$

PROOF OF LEMMA 4
(SUBGRAPH INDUCED)

- Consider the subgraph H_s induced by S and the vertex *(3, 0).*

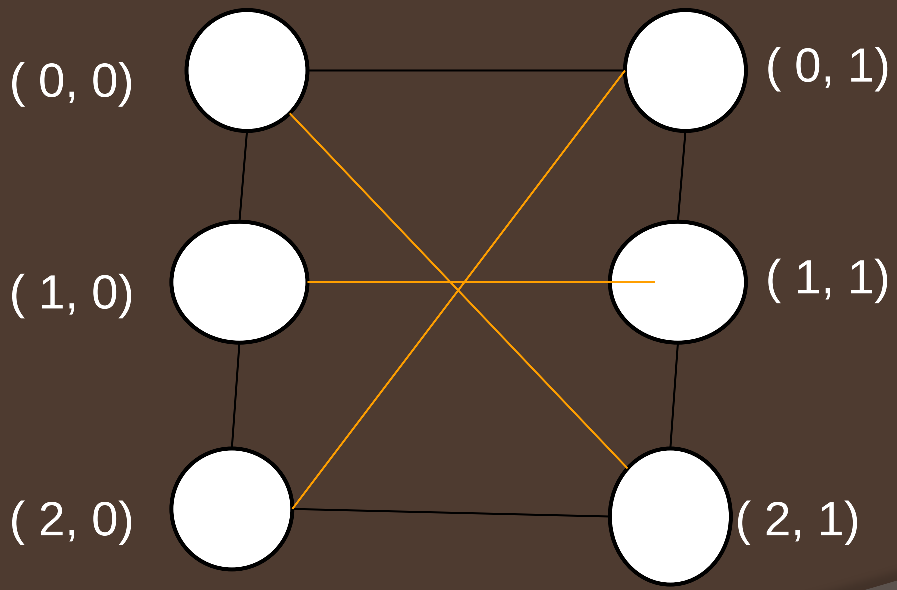


PROOF OF LEMMA 4
(SUBGRAPH INDUCED)

- To satisfy the *co-channel reuse distance* constraint, vertex $(3, 0)$ must get the same color as vertex $(0, 1)$.
- To satisfy the *channel separation* constraint , the colors assigned to vertices $(2, 0)$ and $(3, 0)$ must have a gap of at least 2.

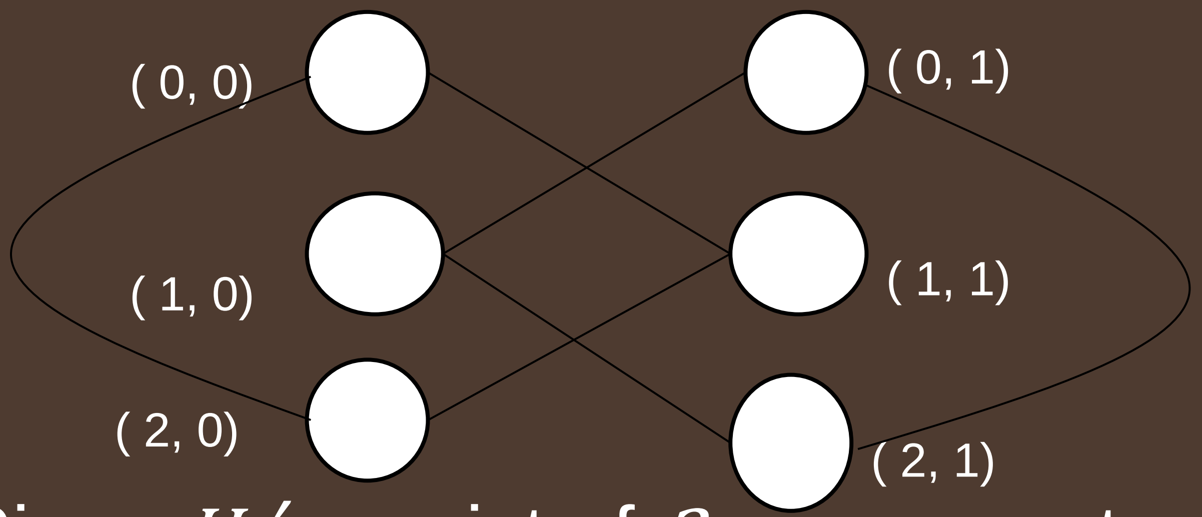
PROOF OF LEMMA 4 (SUBGRAPH INDUCED)

- This is equivalent to add, in H_s, special edges
  $((2, 0), (0, 1)), ((1, 1), (1, 0)), ((2, 1), (0, 0))$



PROOF OF LEMMA 4 (GRAPH COMPLEMENT)

- If we consider H_s' (the complement of graph H_s).



- Since H_s' consist of *2* component connected, it has no hamiltonian path

PROOF OF LEMMA 4 (CONCLUSION)

- From the *Lemma 1*, we can conclude that $\lambda(H) \geq 6$

- Lower bound in the case of $r \geq 5$ and $c = 2$ can be proved by similar arguments

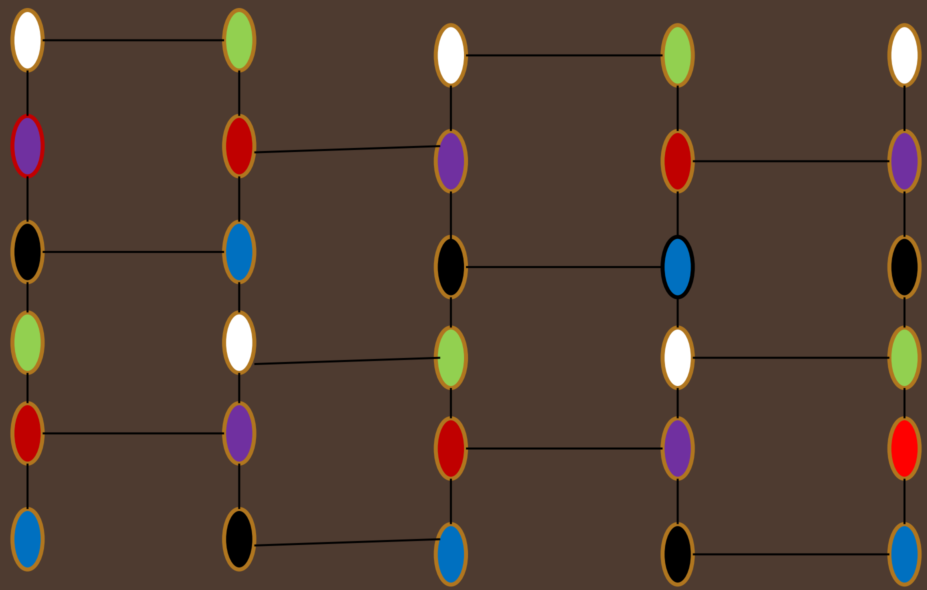
ALGORITHM HEXAGONAL–6–L(2, 1, 1) COLORING

IF (($r \geq 3$) AND ($c \geq 3$)) OR (($r \geq 5$) AND ($c = 2$))

FOR EACH vertex $u = (i, j)$

IF (i MOD 6 = 0 AND j is even) OR (i MOD 6 = 3 AND j is odd)
$f(u) = 0$
IF (i MOD 6 = 0 AND j is odd) OR (i MOD 6 = 3 AND j is even)
$f(u) = 4$
IF (i MOD 6 = 1 AND j is even) OR (i MOD 6 = 4 AND j is odd)
$f(u) = 6$
IF (i MOD 6 = 1 AND j is odd) OR (i MOD 6 = 4 AND j is even)
$f(u) = 2$
IF (i MOD 6 = 2 AND j is even) OR (i MOD 6 = 5 AND j is odd)
$f(u) = 1$
IF (i MOD 6 = 2 AND j is odd) OR (i MOD 6 = 5 AND j is even)
$f(u) = 5$

ALGORITHM HEXAGONAL−6−L(2, 1, 1) COLORING



- White = 0, Black = 1
- Red = 2, Yellow = 3
- Green = 4, Blu = 5, Violet = 6

CORRECTNESS OF THE ALGORITHM

- We have to proof that:
- The *channel separation* constraint is verified
- The *co-channel reuse* constraint is verified

CORRECTNESS (THE CHANNEL SEPARATION CONSTRAINT)

- Let $u=(i, j)$ be a vertex
- For any adjacent v of u such that $v = (i, j+1)$ or $v = (i, j-1)$, it has: $f(v)=f(u)+4$ or $f(v)=f(u)-4$
- Moreover, any pair (u, v) of adjacent vertices on the same coloumn can be colored only in this manners:
  $f(u) = 0$ and $f(v) = 6$
  $f(u) = 6$ and $f(v) = 1$
  $f(u) = 1$ and $f(v) = 4$
  $f(u) = 4$ and $f(v) = 2$
  $f(u) = 2$ and $f(v) = 5$
  $f(u)= 5$ and $f(v) = 0$

CORRECTNESS (THE CHANNEL SEPARATION CONSTRAINT)

- Therefore, a gap between the colors assigned to each pair of adjacent vertices is at least *2*
- So we can conclude that the *channel separation* constraint is verified

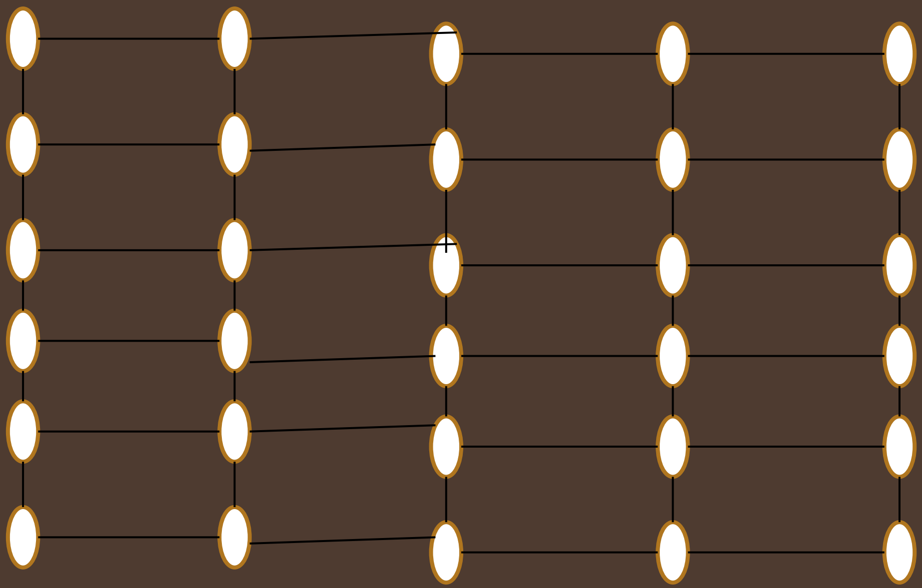
CORRECTNESS (THE CO-CHANNEL REUSE CONSTRAINT)

- Each row of H is colored with 2 colors and any 3 consecutive rows are colored with different colors.
- Vertices (i, j) and $(i, j+1)$ are colored, respectively, as vertices $(i+3, j+1)$ and $(i+3, j)$. Hence, two vertices in rows i and $(i+3)$ get the same color if their distance is at least 4
- The i-th and the $(i+6)$-th rows are colored the same. Hence the same color can be reused only in two vertices at distance 6.

CORRECTNESS (THE CO-CHANNEL REUSE CONSTRAINT)

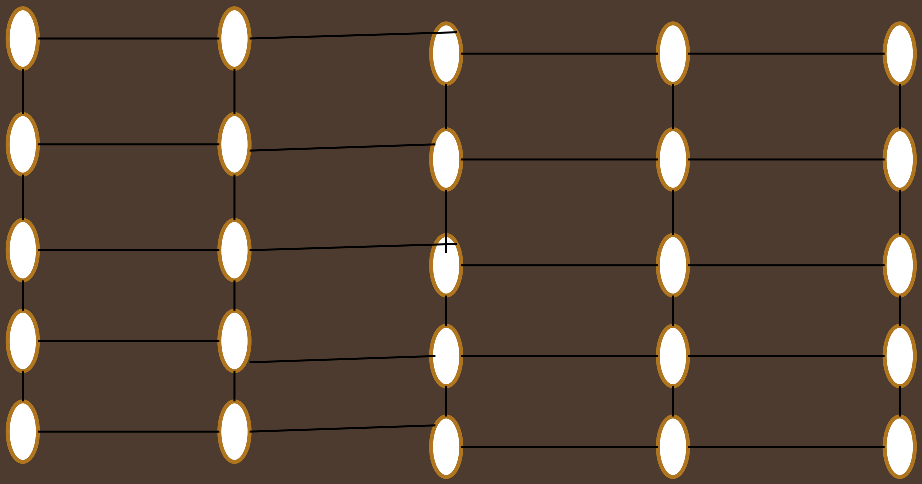
- Finally all the even (and the odd) column are colored in the same way.
- But the distance between vertices (i, j) and $(i, j+2)$ is at least 4, since there are no consecutive horizontal edges.
- So, the *co-Channel Reuse distance* constraint is verified too.

BIDIMENSIONAL GRIDS



- A Bidimensional grid $B(r \cdot c, E)$ is obtained from an hexagonal grid of the same size, simply connecting all the pair of consecutive nodes lying on the same row

BIDIMENSIONAL GRIDS



- A generic vertex (i, j), that is not lying on the board, is adjacent to vertices:
  $(i-1, j), (i+1, j), (i, j-1), (i, j+1)$
- Therefore, a vertex v has degree at most 4

LEMMA 5

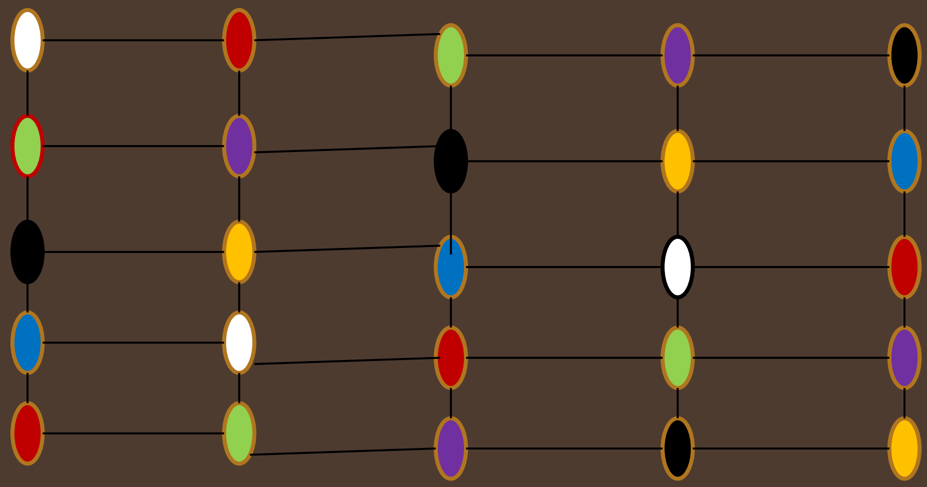
- The optimal *L(2, 1)-coloring* of a bidimensional grid $B(r \cdot c, E)$, where $r \geq 3$ and $c \geq 3$, has $\lambda(B)=6$
- From the *Lemma 2*, since there is at least a vertex with degree 4, we cannot color it with color 0 or 5.

ALGORITHM BIDIMENSIONAL 6-L(2,1) COLORING

IF (($r \geq 3$) AND ($c \geq 3$))

Assign to each vertex $u = (i, j)$ the color $f(u) = (2 \cdot i + 4 \cdot j)$ MOD 7

ALGORITHM BIDIMENSIONAL 6-L(2,1) COLORING



- White = 0, Black = 1
- Red = 2, Yellow = 3
- Green = 4, Blu = 5, Violet = 6

LEMMA 6

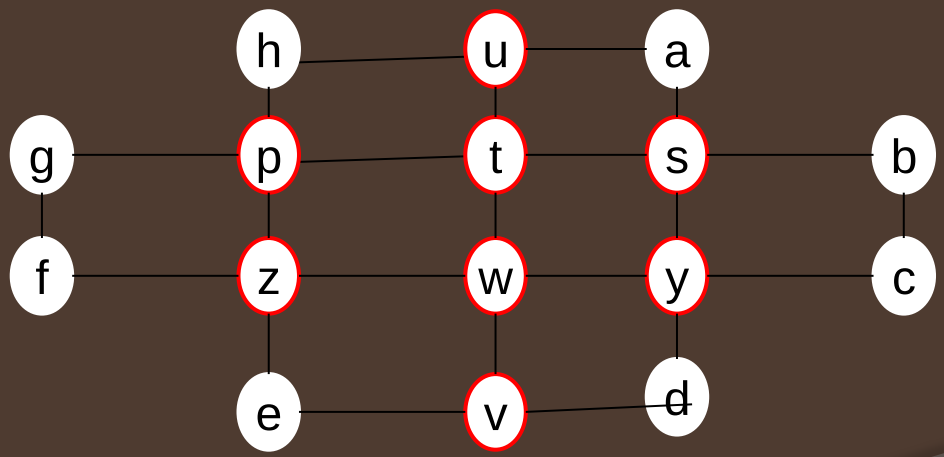
- For $r \geq 5$ and $c \geq 4$, or $r \geq 4$ and $c \geq 5$, there is a *L(2, 1, 1)-coloration* of a bidimensional grid B of size $r \cdot c$ only if $\lambda(H) \geq 8$

PROOF OF LEMMA 6
(CASE r $\geq$ 5 AND c $\geq$ 4)

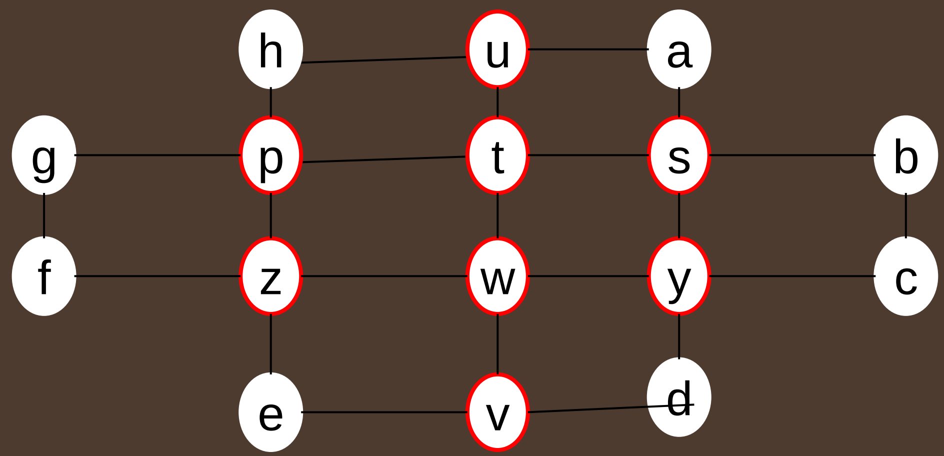
- Let us consider the augmented graph B_4. For any pair of vertices $u = (i, j)$ and $v= (i+3, j)$, let $S_{u,v}$ the following set: $\{ (i, j), (i+1, j), (i+2, j), (i+3, j), (i+1, j-1), (i+2, j-1), (i+1, j+1), (i+2, j+1) \}$
- All of vertices that belongs to $S_{u,v}$ are pairwaise at distance no more than 3
- To satisfy the *co-channel reuse distance constraint* all of those vertices must be colored with different colors, since both $S_{u,v}$ and $S_{u,v}'$ induce a clique in B_4

PROOF OF LEMMA 6

- Now, consider the set:
  $L_{u,v} = S_{u,v}$ U { all the vertices of B at horizontal distance 1 to a vertex on the border of $S_{u,v}$ }



PROOF OF LEMMA 6

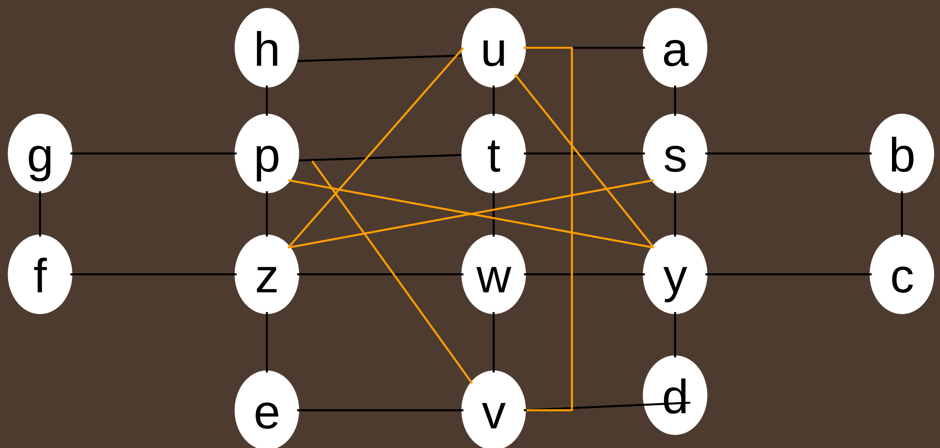


- Let us consider vertices: $a=(i, j+1)$, $b=(i+1, j+2)$ and the bidimensional grid M induced by $S_{u,v}$.
- $S_{u,v}$ has been assigned to all different colors
- If we want to use only 8 colors, vertices b and a must be assigned to the two colors used for the vertices z and v

PROOF OF LEMMA 6

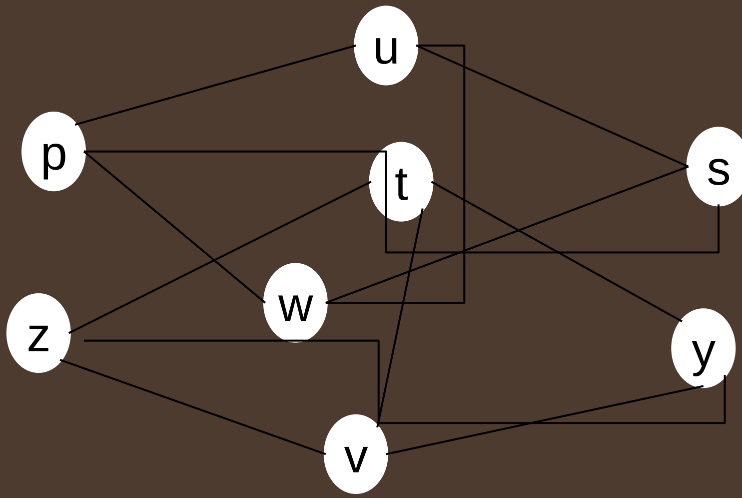
- The color assigned to vertices a and b must be at least 2 from the color assigned to the vertex $s = (i+1, j+1)$.
- This is equivalent to add two edges: (s, z) and (s,v) to the augmented graph.
- Similar arguments we can repeat for the pairs of vertices: $(c, d), (e, f), (g, h)$
- So we can add other edges

PROOF OF LEMMA 6



- Either f or e are colored as vertex u.
- Colors $f(u)$ and $f(v)$ must be assigned to two adjacent vertices in the set $\{ e, f, g, h \}$, in particular $f(u)$ can be assigned to vertex e and $f(v)$ can be assigned to vertex h.
- Thus, one further edge can be added: (u, v)

PROOF OF LEMMA 6



- Let us consider the subgraph M with vertices $\{u, p, t, s, z, w, y, v\}$ and let us build its complement, M'
- Since M' consist of two connected components, M' does not contains an Hamiltonian path

PROOF OF LEMMA 6

- Recalling *Lemma 1* we can conclude that there is no *7-L(2,1,1)-coloring* for a bidimensional grids of size $r \cdot c$, where $r \geq 5$ and $c \geq 4$
- The proof when $r \geq 4$ and $c \geq 5$ is analogous.
- Hence $\lambda(B) \geq 8$

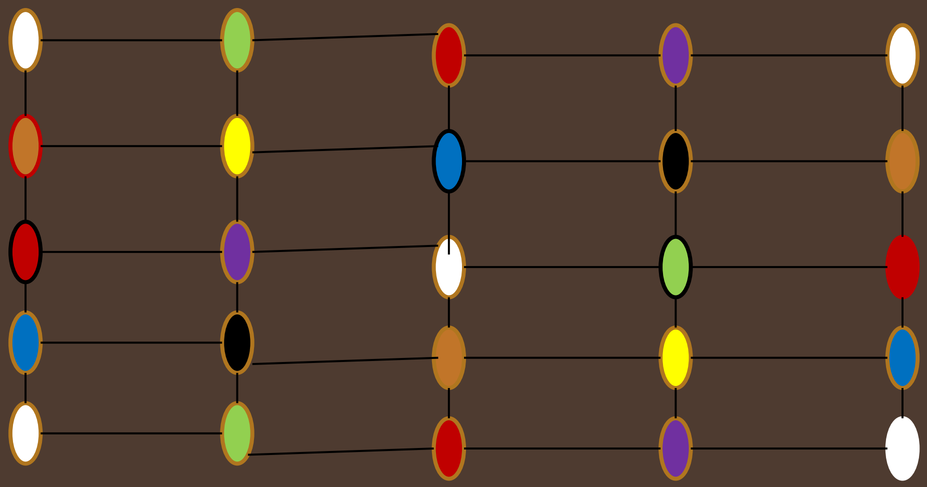
ALGORITHM GRID–8–L(2, 1, 1) COLORING

IF (($r \geq 5$) AND ($c \geq 4$)) OR (($r \geq 4$) AND ($c \geq 5$))

FOR EACH vertex $u = (i, j)$

IF ($(i + j)$ MOD $4 = 0$ AND i is even AND j is even)
$f(u) = 0$
IF ($(i + j)$ MOD $4 = 0$ AND i is odd AND j is odd)
$f(u) = 1$
IF (($i + j)$ MOD $4 = 1$ AND i is even AND j is odd)
$f(u) = 7$
IF ($(i + j)$ MOD $4 = 1$ AND i is odd AND j is even)
$f(u) = 8$
IF ($(i + j)$ MOD $4 = 2$ AND i is even AND j is even)
$f(u) = 2$
IF ($(i + j)$ MOD $4 = 2$ AND i is odd AND j is odd)
$f(u) = 3$
IF ($(i + j)$ MOD $4 = 3$ AND i is odd AND j is even)
$f(u) = 5$
IF ($(i + j)$ MOD $4 = 3$ AND i is even AND j is odd)
$f(u) = 6$

ALGORITHM GRID−8−L(2, 1, 1) COLORING



- White = 0, Black = 1, Red = 2
- Yellow = 3, Azure = 4, Blu = 5
- Violet = 6, Green = 7, Brown = 8

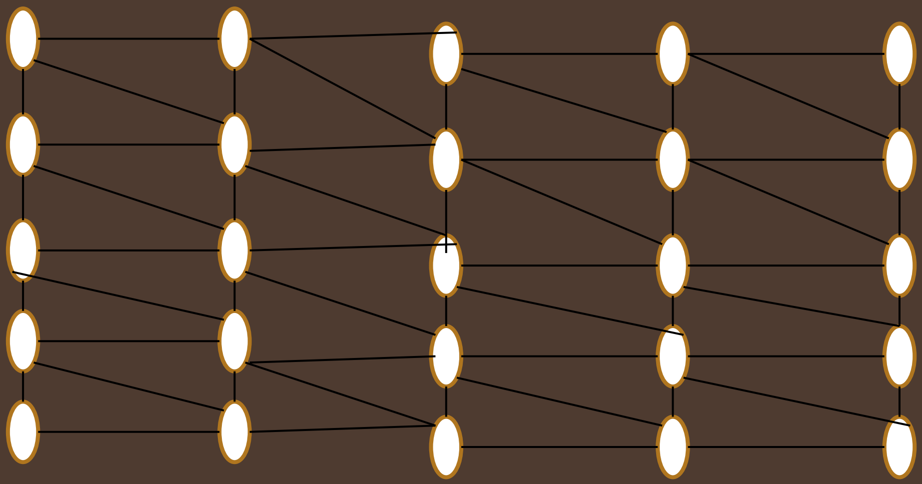
CORRECTNESS OF THE ALGORITHM (CHANNEL SEPARATION CONSTRAINT)

- The channel separation coinstrant is verified by construction of the algorithm
- If color c is assigned to a vertex (i, j), color $c+1$ is assigned to the vertex (i', j'), where:
  - $(i' \bmod 2) \neq (i \bmod 2)$
  - $(j' \bmod 2) \neq (j \bmod 2)$
- Vertices (i, j) and (i', j') are at distance at least 2
- So, any two consecutive vertices cannot be assigned to consecutive colors.

CORRECTNESS OF THE ALGORITHM (CO-CHANNEL REUSE CONSTRAINT)

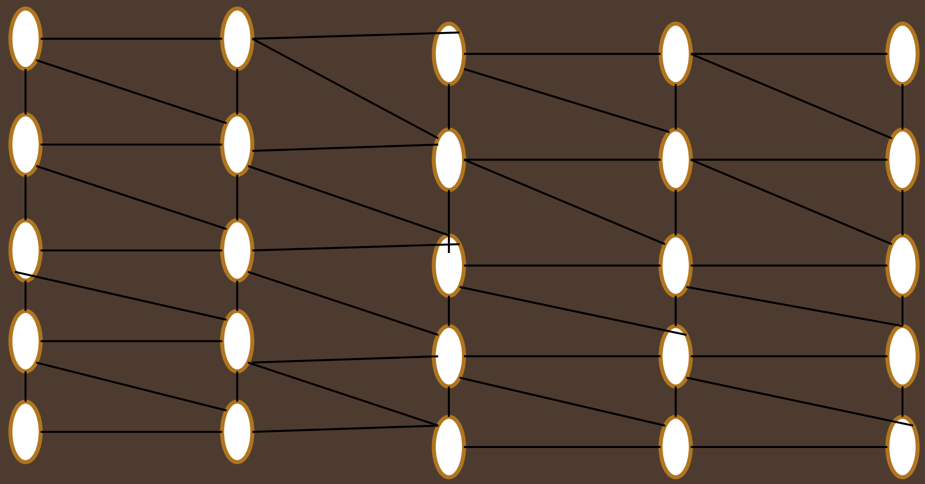
- To verify that the *co-Channel Reuse Distance* constraint is verified it's enough to note that two vertices $u=(i,j)$ and $v=(h,k)$ are assigned to the same color iff :
- $d(u,v)=4$
- Both $/i-h/$ and $/j-k/$ are even.

CELLULAR GRIDS



- A cellular grid C of size $r \cdot c$ is obtained from a bidimensional grid of the same size, augmenting the set of edges with *left-to-right* diagonal connections.

CELLULAR GRIDS



- So, a vertex $u = (i, j)$, that is not lying on the board, is connected with vertices: $(i-1, j)$, $(i+1, j)$, $(i, j-1)$, $(i, j+1)$, $(i-1, j-1)$, $(i+1, j+1)$.
- Therefore it has degree 6

L(2,1) COLORING FOR A CELLULAR GRID

- If one of this condition is verified :
  - $r \geq 5$ and $c \geq 3$
  - $r \geq 3$ and $c \geq 5$
  - $r \geq 4$ and $c \geq 4$
- Then an optimal $L(2,1)$ coloring of a cellular grid C has $\lambda(C) = 8$

ALGORITHM CELLULAR 8-L(2,1) COLORATING

IF (($r \geq 4$) AND ($c \geq 4$)) OR (($r \geq 5$) AND ($c \geq 3$)) OR (($r \geq 3$) AND ($c \geq 5$))

Assign to each vertex $u = (i, j)$ the color $f(u) = (3 \cdot i + 2 \cdot j) \text{ MOD } 9$

ALGORITHM L(2,1,1) COLORING FOR A CELLULAR GRID

- If $r \geq 4$ and $c \geq 4$ an optimal L(2,1,1) coloring of a cellular grid has $\lambda(C) = 11$

```
IF ( ( r ≥ 4) AND ( c ≥ 4 ))

        FOR EACH vertex u = (i, j)

                IF ( (i + j) MOD 6 = 2  AND  i is even AND  j is even )
                        f(u) = 0
                IF ( (i + j) MOD 6 = 0  AND  i is even   AND  j is even)
                        f(u) = 1
                IF (( i + j) MOD 6 = 4  AND  i is even AND  j is even )
                        f(u) = 2
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ALGORITHM L(2,1,1) COLORING FOR A CELLULAR GRID

IF ((i + j) MOD 6 = 1 AND i is odd AND j is even)
f(u) = 3
IF ((i + j) MOD 6 = 3 AND i is odd AND j is even)
f(u) = 4
IF ((i + j) MOD 6 = 5 AND i is odd AND j is even)
f(u) = 5
IF ((i + j) MOD 6 = 5 AND i is even AND j is odd)
f(u) = 6
IF ((i + j) MOD 6 = 2 AND i is odd AND j is odd)
f(u) = 7
IF ((i + j) MOD 6 = 4 AND i is odd AND j is odd)
f(u) = 8
IF ((i + j) MOD 6 = 1 AND i is even AND j is odd)
f(u) = 9
IF ((i + j) MOD 6 = 3 AND i is even AND j is odd)
f(u) = 10
IF ((i + j) MOD 6 = 0 AND i is odd AND j is odd)
f(u) = 11

CONCLUSIONS

	L(2, 1)	L(2, 1, 1)
HEXAGONAL GRIDS	6 Colors	7 Colors
BIDIMENSIONAL GRIDS	6 Colors	9 Colors
CELLULAR GRIDS	9 Colors	12 Colors


That's all Folks!