

# MINIMUM WEIGHT PERFECT MATCHING IN BIPARTITE GRAPHS



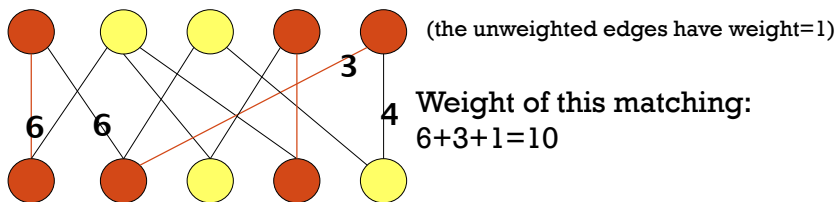
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## WEIGHTED MATCHING (1)

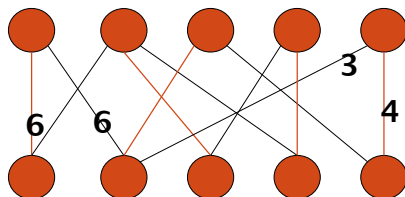
- Each edge has a cost
- The definition of weighted matching is the same as the simple matching (weight does not affect the definition)
- We look for a **minimum weight perfect matching**
- **Note.** This is equivalent to look for a maximum weight perfect matching, where the weights are all negative.

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## WEIGHTED MATCHING (2)



Max weight matching:  
 $6+4+1+1+1=13$

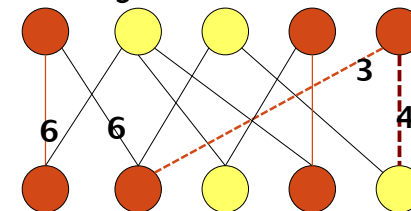


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## WEIGHTED MATCHING (3)

Def. **augmenting path** (different w.r.t. the previous one!) It is any alternating path such that the weight of the edges out of the matching is greater than the weight of the edges in the matching.

**Weight of the augmenting path** = weight of the edges out of  $M$  - weight of the edges in  $M$



**Note.** In this case, aug. paths do not need to end in a free node.

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## WEIGHTED MATCHING (4)

Algorithm:

- Start with an empty matching
- Repeat
  - Find an aug. path  $P$  with max weight
  - If this weight is positive, swap the role of the edges
  - Else return the found matching (that is the one of max weight).
- Complexity: at least  $O(nm)$ .

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## WEIGHTED MATCHING (5)

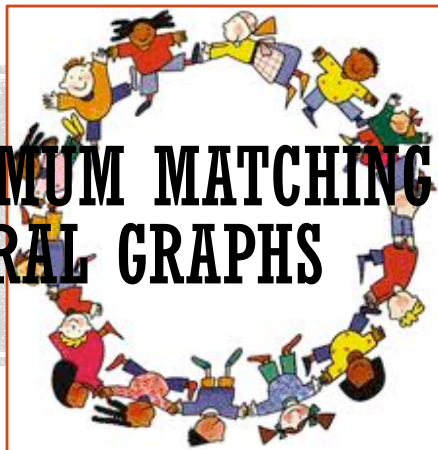
- It is possible to model the minimum weight perfect matching problem as an ILP problem (**Hungarian method**):
  - Given a matching  $M$ , let  $x$  be its incidence matrix, where  $x_{ij} = 1$  if  $(i, j)$  is in  $M$  and  $x_{ij} = 0$  otherwise.
  - The problem can be written as follows:

$$\text{minimize } \sum_{i,j} c_{ij}x_{ij} \quad \text{subject to } \sum_j x_{ij} = 1, i \in A$$

- Complexity:  $O(n^3)$ .
- $$\sum_i x_{ij} = 1, j \in B$$
- $$x_{ij} \geq 0, i \in A, j \in B$$
- $$x_{ij} \text{ integer}, i \in A, j \in B$$

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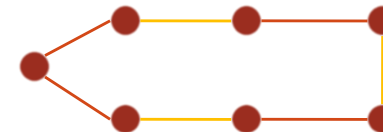
## MAXIMUM MATCHING IN GENERAL GRAPHS



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## BLOSSOMS (1)

- We have already noticed that the critical point of general graphs are odd cycles containing a maximal number of edges in the matching



- Such cycles are called **blossoms**

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## BLOSSOMS (2)

▪ **Lemma (cycle contraction).** Let  $M$  be a matching of  $G$  and let  $B$  be a blossom. Let  $B$  node-disjoint from the rest of  $M$ . Let  $G'$  be the graph obtained by  $G$  contracting  $B$  in a single node. Then  $M'$  of  $G'$  induced by  $M$  is maximum in  $G'$  iff  $M$  is maximum in  $G$ .

▪ **Proof.**  $M$  max in  $G \Rightarrow M'$  max in  $G'$

By contradiction. Assume  $M'$  not max. Hence there exists an aug. path  $P$  in  $G'$  w.r.t.  $M'$ .

Let  $b$  be the node representing  $B$ .

Two cases can hold:

1.  $P$  does not cross  $b \Rightarrow P$  augmenting for  $M$ , too.  
Contradiction

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## BLOSSOMS (4)

Proof of the Cycle contraction lemma – cntd.

▪  $M'$  max in  $G' \Rightarrow M$  max in  $G$

By contradiction,  $M$  is not max. Let  $P$  be an aug. path for  $M$ .

Two cases hold:

1.  $P$  does not cross  $b \Rightarrow P$  is aug. for  $G'$ . A contradiction.

2.  $P$  crosses  $b$ . Since  $B$  contains only one free node, at least an end-point of  $P$  lies outside  $B$ . Let it be  $w$ .

Let  $P'$  be the sub-path of  $P$  joining  $w$  with  $b$ .

$P'$  is an aug. path for  $G'$ . A contradiction. ■

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## BLOSSOMS (3)

Proof of the Cycle contraction lemma – cntd.

Observe that  $b$  is free as it represents the node  $v$  in  $B$  adjacent to two edges out of  $M$ . In other words,  $v$  is free if we restrict to  $B$ .

2.  $P$  crosses  $b \Rightarrow b$  must be an end-point of  $P$ .

Define  $P' = P \cup P''$  where  $P''$  is inside  $B$ .

$P'$  is augmenting for  $G$ . A contradiction.

...

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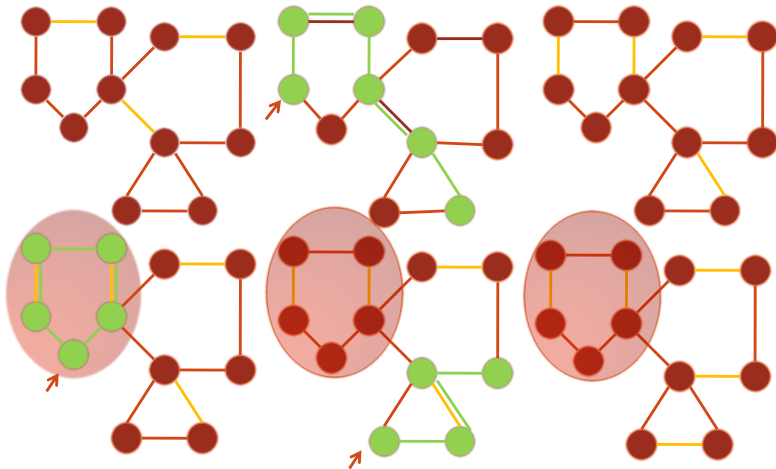
## MAX MATCHING IN GENERAL GRAPHS (1)

- In order to find an aug. path in general graphs, it is “enough” to modify the algorithm on bipartite graphs in order to include blossom search.
- For each found blossom, it is shrunk in a node and a new (reduced) graph is generated.
- Each aug. path found in this new graph can be easily “translated” into an aug. path in  $G$ .
- Thanks to the previous lemma, if  $M$  is max in the new graph, it is max even in  $G$ .
- This is the Edmonds algorithm [‘65]
  - The time complexity depends on how blossoms are handled. Varying with the used data structures, it can be either  $O(n^3)$  or  $O(mn^2)$ . The best known time complexity is  $O(m\sqrt{n})$  [Micali & Vazirani ‘80]

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## MAX MATCHING IN GENERAL GRAPHS (2)

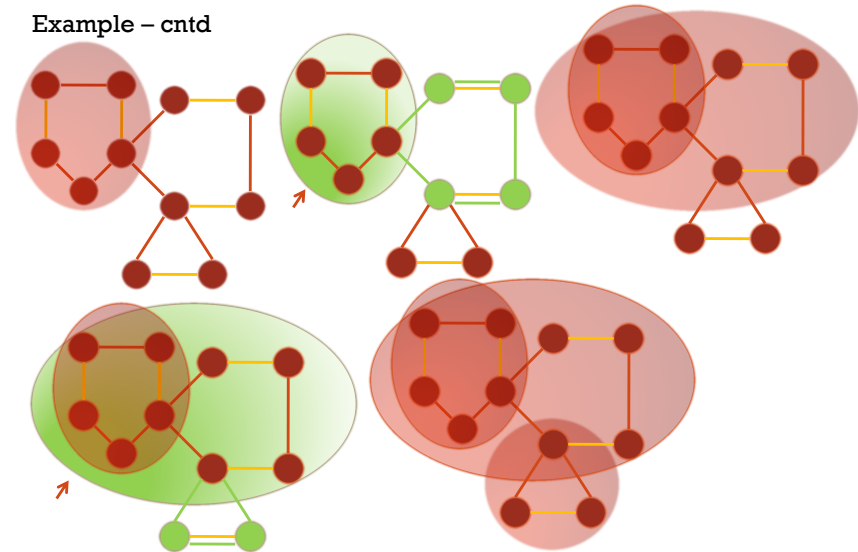
Example:



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## MAX MATCHING IN GENERAL GRAPHS (3)

Example – cntd




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## MAX MATCHING IN GENERAL GRAPHS (4)

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Edmonds Algorithm ['65]

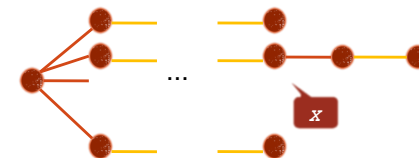
- $M$  matching for  $G$
- $L$  subset of the free nodes (if  $L$  empty  $\Rightarrow M$  max)
- $F$  forest s.t. each node of  $L$  is the root of a tree in  $F$
- Expand  $F$  by adding 
- Nodes that are at odd distance from a node of  $L$  have degree 2 (1 in  $M$  and 1 in  $E \setminus M$ ): we call them **internal nodes**
- The other nodes: **external nodes**
- ...

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## MAX MATCHING IN GENERAL GRAPHS (5)

Edmonds algorithm – cntd

- Consider the neighbors of the external nodes.
- 4 possibilities hold:
  1. There exists  $x$  external and incident to a node  $y$  not in  $F$ :  
add to  $F$  edges  $(x,y)$  and  $(y,z)$ , and  $(y,z)$  is in  $M$ .

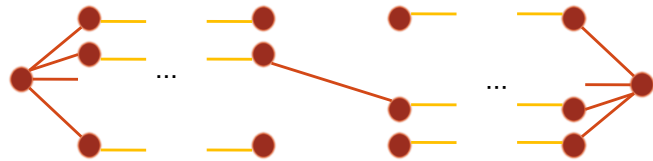


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## MAX MATCHING IN GENERAL GRAPHS (6)

Edmonds algorithm – cntd

2. Two external nodes lying in two different components of  $F$  are adjacent:  
augmenting path



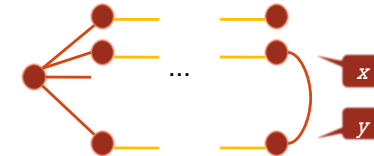
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## MAX MATCHING IN GENERAL GRAPHS (7)

Edmonds algorithm – cntd

3. Two external nodes  $x, y$  in the same component in  $F$  are adjacent:

let  $C$  be the found cycle. It is possible to move the edges in  $M$  around  $C$  so that the cycle contraction lemma can be used  $\Rightarrow$  reduced graph  $G'$

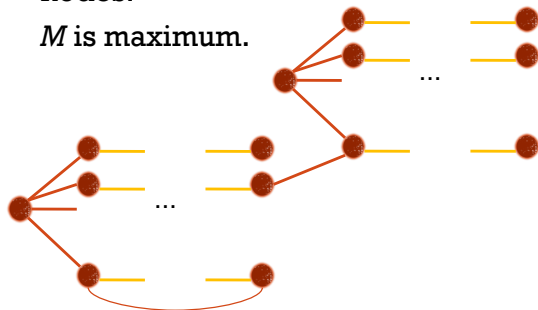


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## MAX MATCHING IN GENERAL GRAPHS (8)

Edmonds algorithm – cntd

4. All the external nodes are adjacent to internal nodes:  
 $M$  is maximum.



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## MAX MATCHING IN GENERAL GRAPHS (9)

**Lemma.** At each step of the Edmonds algorithm, either the dimension of  $F$  increases, or the dimension of  $G$  decreases, or an aug. path is found, or  $M$  is maximum.

**Complexity.** Number of iterations  $\leq$   
 num. of times  $F$  is increased (at most  $n$ ) +  
 num. of times a blossom is shrunk (at most  $n$ ) +  
 num. of found aug. paths (at most  $n/2$ ).

The time complexity depends on how blossoms are handled. Varying with the used data structures, it can be either  $O(n^3)$  or  $O(mn^2)$ .

Best known time complexity:  $O(m\sqrt{n})$

[Micali & Vazirani '80]

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## ANOTHER APPLICATION



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## SWITCH BUFFER (2)

- The  $\log N$ -stage networks are not rearrangeable, i.e. not all routes can be done simultaneously
- Two packets may want to use the same link at the same time
- **Solution:** buffering (though buffers increase delay)

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## SWITCH BUFFER (1)

### Reminder:

- Interconnection topologies are constituted by layers of basic modules that are  $2 \times 2$  cross-bar switches
- Any output can be reached by any input by properly setting some switches
- A single routing can be easily performed if the network is self-routing (e.g. Butterfly, Baseline, etc.)

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## MULTISTAGE TOPOLOGIES WITH BUFFERS (1)

The multistage topologies are good to use, because they are:

- modular
- scalar

Nevertheless, the buffers at each node provoke:

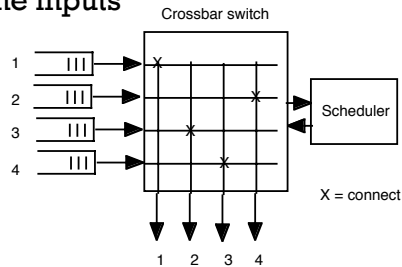
- delays for going through the stages
- decreased throughput due to internal blocking

**Solution:** (input) buffers that are external to the topology

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## MULTISTAGE TOPOLOGIES WITH BUFFERS (2)

- **Head of line (HOL) buffer:** only the first packet can leave the buffer.
- Buffers are connected through a crossbar network to the inputs of the topology
- During each slot, the scheduler establishes the crossbar connections to transfer packets from the buffers to the inputs



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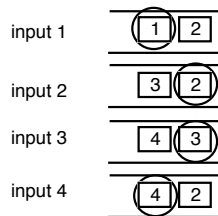
## MULTISTAGE TOPOLOGIES WITH BUFFERS (3)

- When the packets at the head of two or more input queues are destined to the same input node, only one can be transferred and the other is blocked
- This behavior limits throughput because some inputs (and consequently outputs) are kept idle during a slot even when they have other packets to send
- ...

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## MULTISTAGE TOPOLOGIES WITH BUFFERS (4)

- If the inputs are allowed to transfer packets that are not at the head of their buffers, throughput can be improved
- **Example:**



- How does the scheduler decide which input to transfer to the network?

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## MULTISTAGE TOPOLOGIES WITH BUFFERS (5)

**Backlog matrix:**

- rows: input buffers
- columns: outputs
- each entry  $(i,j)$  represents the number of packets in buffer  $i$  destined to output  $j$

		output		
		1	2	3
input	1	3	3	0
	2	2	0	0
	3	0	0	2

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## MULTISTAGE TOPOLOGIES WITH BUFFERS (6)

- During each slot, the scheduler can transfer at most one packet from each buffer to each output
- The scheduler must choose at most one packet from each row and from each column of the backlog matrix
- This can be done by solving a bipartite matching algorithm...

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## MULTISTAGE TOPOLOGIES WITH BUFFERS (8)

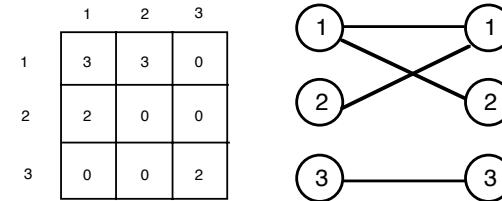
- Finding a maximum matching during each time slot does not eliminate the effects of HOL blocking
- It is, indeed, necessary to look beyond a single slot when making scheduling decisions
- **Solution:** edge  $(i,j)$  is assigned a weight equal to the value of element  $(i,j)$  of the backlog matrix
- **Theorem:** A scheduler that chooses, during each time slot, the maximum weighted matching achieves full utilization.
- Proof and other details: see [McKeon et al. 1999]

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## MULTISTAGE TOPOLOGIES WITH BUFFERS (7)

- The bipartite graph  $G=(V \cup W, E)$  is built as follows:
- $V$ :  $N$  nodes representing the buffers
- $W$ :  $N$  nodes representing the outputs
- $E$ : there is an edge from a buffer  $i$  to an output  $j$  iff there is a packet in the backlog matrix to be transferred from  $i$  to  $j$ .

▪ **Example:**



- Finding a maximum matching is equivalent to finding the largest set of packets that can be transferred simultaneously

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