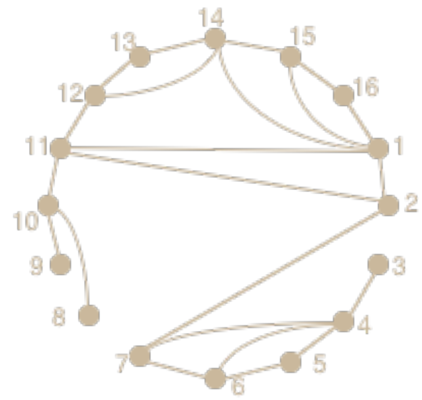
APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(1)

- $\lambda_{2,1}(G) \leq 2\Delta+4$ because G has treewidth 2
- Jonas ['93]: $\lambda_{2,1}(G) \leq 2\Delta+2$
- Bodlaender et al. ['04]: $\lambda_{2,1}(G) \leq \Delta+8$ but they conjecture that $\lambda_{2,1}(G) \leq \Delta+2$ -> possible students' lesson
- C.& Petreschi ['04] $\Delta+1 \leq \lambda_{2,1}(G) \leq \Delta+2$ and they conjecture that this algorithm gives the optimum value.

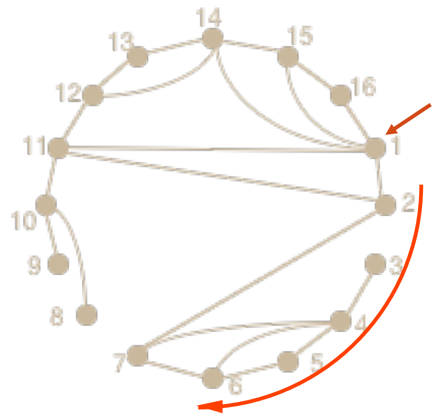
APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(2)

Def. A graph is said to be outerplanar if it can be represented as a plane graph so that each node lies on the border of the external face

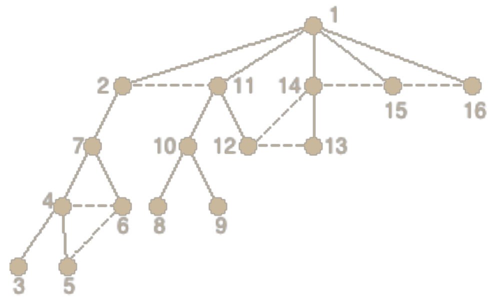


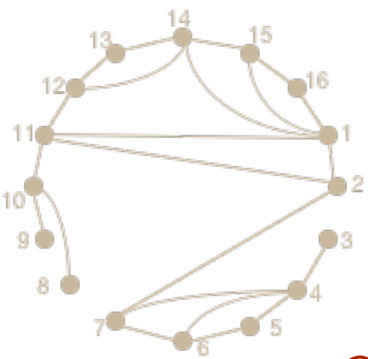
APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(3)

- Ordered Breadth First Tree:
- Choose a node r
- Induce a total order on the nodes on the external face
- Run a BFS from r so that nodes coming before in the ordering are visited before than the others.



APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(4)





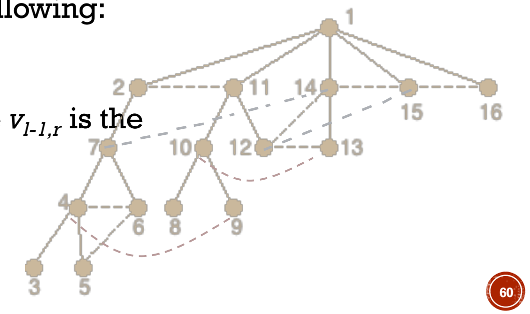
APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(5)

Th. (well known)

$G=(V,E)$; BFT $T=(V,E')$

Each non tree edge $(v_{l,h}, v_{l',k})$ satisfies one of the following:

- $l'=l$
- $l'=l-1$ and $r<k$, where $v_{l-1,r}$ is the parent of $v_{l,h}$.

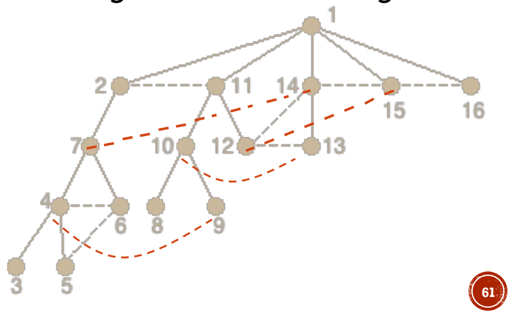


APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(6)

Th. $G=(V,E)$; OBFT $T=(V,E')$:

- If $(v_{l,h}, v_{l,k})$, $h<k$, then $k=h+1$
- If $v_{l,h}$ is a child of $v_{l-1,i}$ and $(v_{l,h}, v_{l-1,k})$ is a non tree edge, $i<k$, then $k=i+1$ and $v_{l,h}$ is the rightmost of its siblings

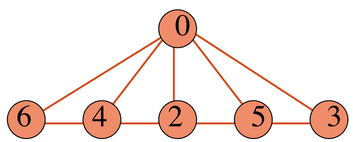
(orange edges are not admissible)

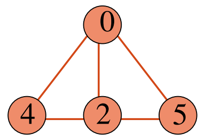


APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(7)

Lemma. If $\Delta\geq 4$, $\Delta+2$ colors are necessary.

Proof.



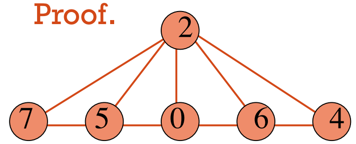


$\boxed{0\ \ 1}$ 2 3 4 5 6

APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(8)

- Lemma. If the root of the tree has an already assigned color, then $\Delta+3$ colors are necessary.

Proof.



0 $\boxed{1\ \ 2\ \ 3}$ 4 5 6 7

General technique

APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(9)

- INPUT: G outerplanar with max degree Δ
- OUTPUT: a feasible $L(2,1)$-labeling for G
- Consider a node v with max degree and run an OBFS starting from v
- $Label(v) \leftarrow 0$
- Label the first layer according to the previous Lemma
- For each layer $l \geq 2$, top-down, from left to right

  repeat

  Label the children of $v_{l,k}$ according to the previous Lemma, eliminating from the palette the forbidden colors

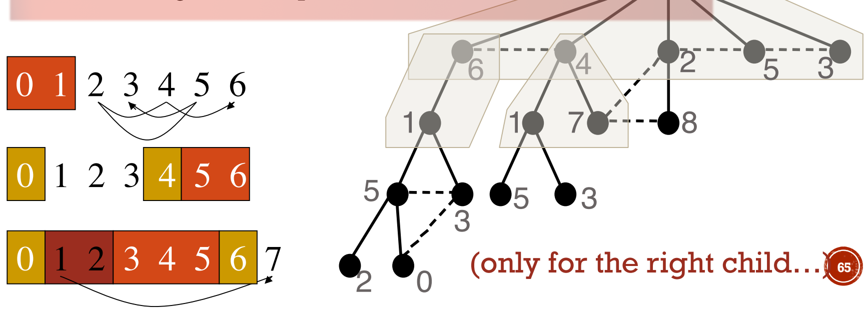
APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(10)

An example:

- INPUT: G outerplanar with max degree Δ
- OUTPUT: a feasible $L(2,1)$-labeling for G
- Consider a node v with max degree and run an OBFS starting from v
- $Label(v) \leftarrow 0$
- Label the first layer according to the previous Lemma
- For each layer $l \geq 2$, top-down, from left to right

  repeat

  Label the children of $v_{l,k}$ according to the previous Lemma, eliminating from the palette the forbidden colors



APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(11)

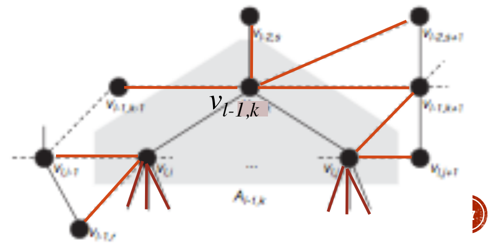
Correctness and Bounds

- Best previously known results: $\lambda \leq \Delta+8$ for each Δ [BKTL'00]
- Conjecture [BKTL00]: $\lambda \leq \Delta+2$ for each Δ
- Th. The provided algorithm correctly $L(2,1)$-labels each outerplanar graph with max degree $\Delta \geq 8$ with $\lambda \leq \Delta+2$ in linear time; otherwise, at most 11 colors are anyway necessary.
- Proof. By induction, considering the edges coming out from the subgraph induced by any node and its children…

APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(12)

- a. one tree-edge to the father of $v_{l-1,k}$
- b. at most three non-tree edges connecting $v_{l-1,k}$ with some nodes either at the same layer or at the previous layer
- c. at most two non-tree edges from the leftmost sibling
- d. at most two non-tree edges from the rightmost sibling

The other outgoing edges do not contribute to the labeling during this step.

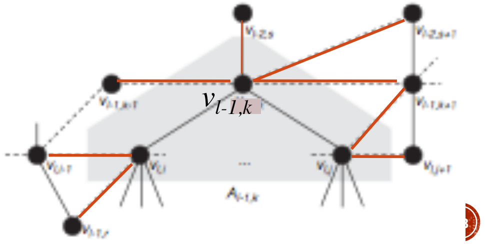


APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(13)

(Proof sketch) By inductive hypothesis at most $\Delta+3$ colors have been used. We prove that they are sufficient to label the children of $v_{l-1,k}$.

We cannot use:

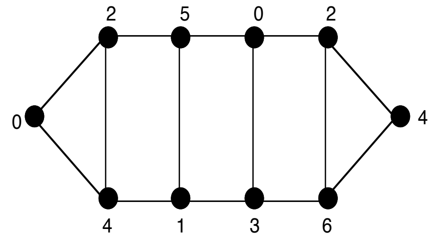
- At most 3 colors (for $v_{l-1,k}$) $\rightarrow \Delta$
- At most 1 color (edge a) $\rightarrow \Delta-1$
- At most x colors (edges b), $0 \leq x \leq 3$ $\rightarrow \Delta-1-x$ for the $\Delta-1-x$ children of $v_{l-1,k}$
- The c edges give conditions only on 1 or 2 nodes, hence it is possible to arrange
- Analogously for the d edges



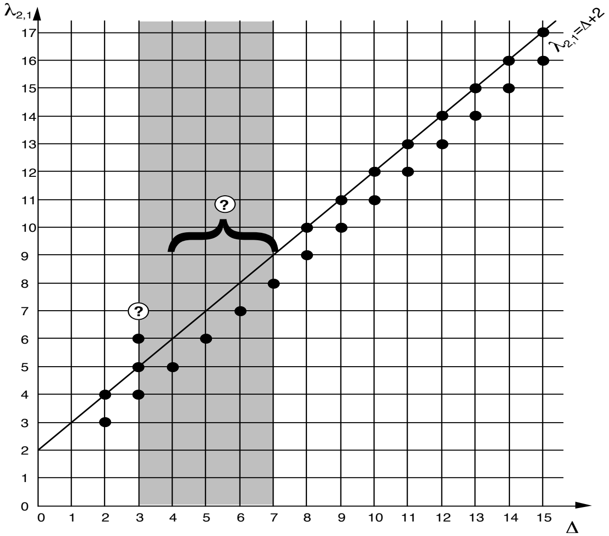
APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(14)

The special case $\Delta=3$

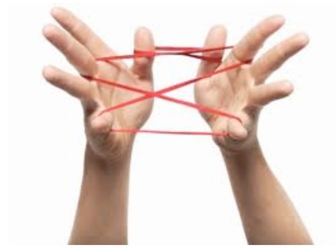
- There exists an infinite class of outerplanar graphs having $\Delta=3$ requiring $\lambda=\Delta+3$
- It is possible to provide a labeling algorithm for these graphs using $\lambda \leq \Delta+5$



APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(15)



VARIATIONS OF THE PROBLEM



VARIATIONS OF THE PROBLEM (1)

ORIENTED L(2,1)-LABELING

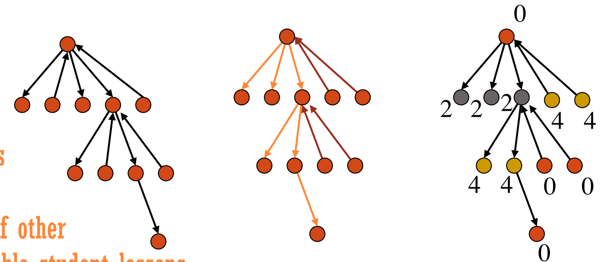
- An oriented L(2,1)-labeling of a directed graph *G* is a function assigning colors from 0, … ,λ to the nodes of *G* so that nodes at distance 2 in the graph take different colors and adjacent nodes take colors at distance 2.
- Oriented L(2,1)-labeling problem
  minimizing λ
- Note. The minimum value of λ can be very different from the value of the same parameter in the undirected case. Example: trees…

VARIATIONS OF THE PROBLEM (2)

ORIENTED L(2,1)-LABELING

- Reminder: In undirected trees, $\Delta+1 \leq \lambda \leq \Delta+2$, and the exact value is linearly decidible

  [Chang & Kuo '96, Hasunama et al. 2008]
- In directed trees, $\lambda \leq 4$ [Chang & Liaw '03]



Available papers on the oriented L(2,1)-labeling of other graphs for possible student lessons…

VARIATIONS OF THE PROBLEM (3)

$L(h_1, \dots, h_k)$-LABELING

With the aim of making the model more realistic:

- An $L(h_1, \dots, h_k)$-labeling of a graph *G* is a function assigning integer values to the nodes of the graph such that:

  $|l(u)-l(v)| \geq h_i$ if *u* and *v* are at distance *i* in the graph, $1 \leq i \leq k$.
- $L(h_1, \dots, h_k)$-labeling problem: minimizing λ
- Particularly interesting: $L(2,1,1)$ and $L(\delta, 1, \dots, 1)$.
- Even these special cases are NP-hard on general graphs, so special classes of graphs are handled.

VARIATIONS OF THE PROBLEM (4)

BACKBONE COLORING

If the topology has a backbone, where the transmitting power is higher wrt the rest of the network:

- A *Backbone coloring* of a graph *G* wrt a graph *H* is a function assigning integer values to the nodes of the graph such that:

  $|l(u)-l(v)| \geq 2$ if (u,v) is an edge of *H* and

  $|l(u)-l(v)| \geq 1$ if (u,v) is an edge og *G-H*.
- *Backbone coloring problem*:

  minimizing λ

Available papers on this coloring for student lessons…

VARIATIONS OF THE PROBLEM (5)

n-MULTIPLE L(*h,k*)-LABELING

In practice, each transmitting station is able to handle more than one channel, so a set of channels is assigned to it.

- Given two set of integer values I and J, we define

  $dist(I,J)=min\{|i-j|: i \text{ in } I \text{ and } j \text{ in } J\}$

Example:

$I=\{0,1,2\}; J=\{4,5,6\}; dist\{I,J\}=2.$

VARIATIONS OF THE PROBLEM (6)

n-MULTIPLE L(*h,k*)-LABELING

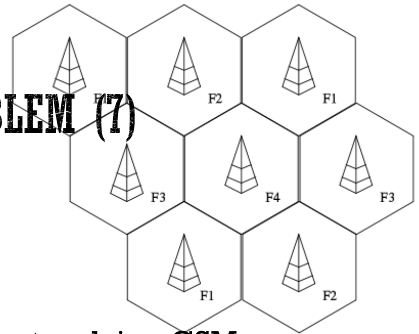
- An *n-multiple L(h,k)-labeling* of a graph G is a function assigning n integer values to each node of the graph so that:

  $dist(l(u),l(v)) \geq h$ if (u,v) is an edge of G and

  $dist(l(u),l(v)) \geq k$ if u and v are at dist. 2 in G.

- *n-multiple L(h,k)-labeling problem*:

  minimizing λ, given n.



VARIATIONS OF THE PROBLEM (7)

FREQUENCY ASSIGNMENT IN A GSM NETWORK

In the special case in which the network is a GSM:

- The network is a cellular network with hexagonal cells.
- Each cell has its own station connecting the fixed network devices with the mobile devices that are at moment inside the cell.
- Mobile phones connect to the GSM network trying to communicate with the station associated to the cell where they lie.

VARIATIONS OF THE PROBLEM (8)

FREQUENCY ASSIGNMENT IN A GSM NETWORK

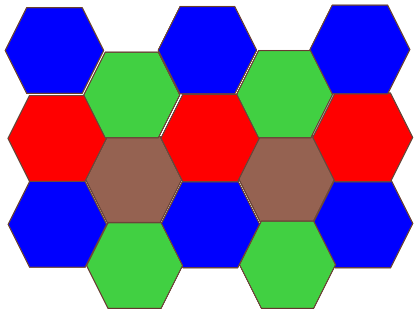
In a GSM network, the cells need to use different frequences, in order not to interfere.

- Coloring map problem: given a plane map, the problem consists in coloring each region in such a way that adjacent regions take different colors and that the min number of colors is used.
- Four Color Theorem: It is always possible to color a map using at most 4 colors.

VARIATIONS OF THE PROBLEM (9)

FREQUENCY ASSIGNMENT IN A GSM NETWORK

- It follows that 4 different frequencies are sufficient for an arbitrary GSM network:

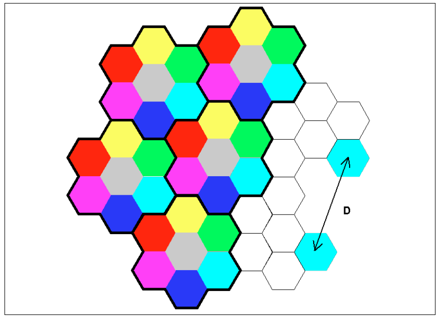


VARIATIONS OF THE PROBLEM (10)

FREQUENCY ASSIGNMENT IN A GSM NETWORK

- In fact, more sofisticated variations of this problem lead to 7 colors -> L(*h*,*k*)-labeling again

More in detail…



VARIATIONS OF THE PROBLEM (11)

FREQUENCY ASSIGNMENT IN A GSM NETWORK

- In wireless communication networks of the 1st and 2nd generation, the concept of cellular channel allocation and spatial frequency reuse were the key ideas that have driven the initial success of mobile telephony.
- For example, a seven color labeling of a hexagonal grid was on the basis of the AMPS (American Mobile Phone System).
- The same scheme existed for GSM.
- …

VARIATIONS OF THE PROBLEM (12)

FREQUENCY ASSIGNMENT IN A GSM NETWORK

- In the 3rd generation of mobile systems, the introduction of CDMA (Code Division Multiple Access) has enabled the reuse of the whole frequency band in each cell: instead of dividing the signal space in time or frequency, a code of pseudorandom sequence is used to differentiate the signal from each transmitter.
- In this context, the labeling schemes were of much reduced importance.
- …

VARIATIONS OF THE PROBLEM (13)
FREQUENCY ASSIGNMENT IN A GSM NETWORK

- The 4th generation mobile standards mainly use Orthogonal Frequency Division Multiple Access schemes.
- These schemes divide the signal space in time slots and orthogonal frequencies.
- At the middle of a cell, all slots of time and frequencies are allocated to users. At the edge of a cell, only part of the band is used and a three color scheme is used.
- Even this model can be reduced to a labeling scheme. For further details:

[Archetti, Bianchessi, Hertz, Colombet, Gagnon '13]

A PARENTHESIS ON THE 4 COLOR PROBLEM (1)

- Given a map, it can be naturally considered a planar graph G.
- Given G, let G^* its dual graph:
  - Put a node of G^* in each region of G
  - Connect two nodes of G^* iff the corresponding regions (faces) are adjacent (i.e. share an edge in G)
- A vertex coloring of G^* corresponds to a map coloring of G.

A PARENTHESIS ON THE 4 COLOR PROBLEM (2)

- In fact, cartographers have always known that 4 colors were enough for each kind of map, but in 1852 Francis Guthrie wondered whether this fact could be proved.
- After more than 100 years, and many (wrong) announcements, Appel and Haken proved the 4 Color Theorem in 1976.
- The complete proof is computer assisted because it exhaustively examines more than 1700 configura- tions.
- More recently, Robertson, Sanders, Seymour, and Thomas wrote a new proof, needing to examine "only" 633 configurations.

A PARENTHESIS ON THE 4 COLOR PROBLEM (3)

There are some interesting results for other numbers of colors:

- 2-coloring.
  Polynomially solvable:
  - Assign a color to a region.
  - Assign the other color to its neighbor regions.
  - Assign the first color to its neighbor regions.
  - Continue until the regions have been all colored or there is a color conflict. In this latter case the map is not 2-colorable.

A PARENTHESIS ON THE 4 COLOR PROBLEM (4)

- 3-coloring
  - NP-hard, hence no algorithms to decide whether a map is 3-colorable or not.
  - Method: exhaustively try all the color combinations for the regions.
  - Inapplicable: for N regions, there are 3^N possibilities. (if $N=48$ the combinations are about 8×10^{22})
  - …

A PARENTHESIS ON THE 4 COLOR PROBLEM (5)

- 3-coloring (cntd)
  - There are some techniques in order to simplify the map before coloring it (for example, if a region has only 2 neighbor regions, it can be eliminated from the map: when it is re-inserted, it will be colored with the third color) but the worst case time complexity is the same.

A PARENTHESIS ON THE 4 COLOR PROBLEM (6)

- 4-coloring
  - The proof of the 4 color theorem is constructive, and so it shows how to find a feasible coloring, but the number of cases is too high to be useful in practice.
  - There are some transformations, similar to those used for the 3-coloring, but they do not eliminate the need of exhaustively try all the possibilities.

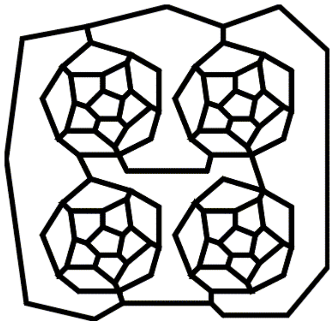
A PARENTHESIS ON THE 4 COLOR PROBLEM (7)

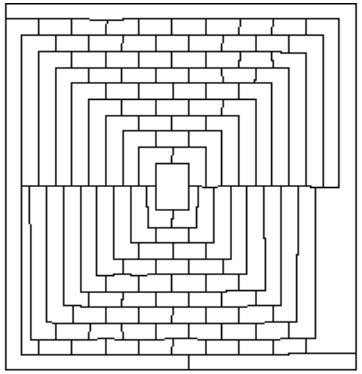
- 5-coloring
  - It is relatively easy to color a map using 5 colors. There is an algorithm that first simplifies the map eliminating all the regions and then re-insert them assigning the correct color.

A PARENTHESIS ON THE 4 COLOR PROBLEM (8)

We conclude with a puzzle:

Try to 4-color these 2 maps…





In 1975 Martin Gardner claimed he could prove that this map was not 4-colorable (April fool)

A PARENTHESIS ON THE 4 COLOR PROBLEM (9)

Solutions