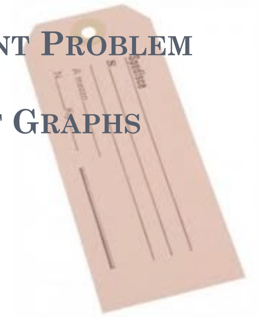
SECOND PART:
WIRELESS NETWORKS
2.A. AD HOC NETWORKS

THE FREQUENCY ASSIGNMENT PROBLEM
I.E.
THE *L(h,k)*-LABELING OF GRAPHS



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Network Algorithms
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THE PROBLEM

THE PROBLEM (1)

There is not "the" frequency assignment problem.

Frequency assignment is necessary in many different types of wireless networks.

Depending on the particular network, the understanding of frequency assignment varies.

For this reason, several "flavors" of frequency assignment are present in the literature.

The Problem (2)

Wireless communication between two points is established with the use of a transmitter and a receiver.

The transmitter generates electrical oscillations at a radio frequency.

The receiver detects these oscillations and transforms them into sounds or images.

When two transmitters use the same frequency, they may interfere.

The Problem (3)

One of the most popular applications of wireless communication is the establishment of fixed cellular telecommunication networks.

In contrast to mobile cellular networks, in non-mobile or fixed systems both the transmitters and the receivers are located at fixed points in the area of interest.

Fixed cellular networks provide a financially attractive alternative to the construction of conventional wired networks.

…

The Problem (4)

Moreover, the introduction of new services, such as data communication (internet, e-mail) and video-conferencing cause shortage of capacity in existing wired networks.

Point-to-point wireless connections can be used as an alternative to the extension of the capacity of these wired networks.

In both cases no cable connections have to be established.

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The Problem (5)

A disadvantage of point-to-point connections is that the transmitter and receiver have *to see* each other, which means that there should be no obstacles in between them.

As a consequence, transmitters and receivers have to be built at high locations (e.g., at the roof of apartment and office buildings).

Although the transmitters are directed to the receivers, their signals can interfere. Especially if signals cross each other, the use of (almost) the same frequencies should be avoided.

…

The Problem (6)

Another application that has similarities with fixed cellular networks stems from the military.

In military communication networks, wireless connections have to be established between pairs of transceivers.

These connections, or radio links, can interfere with each other, if they use similar frequencies in the same area.

The Problem (7)

The rapid development of new wireless services (e.g. digital cellular phone networks) resulted in a run out of the most important (and expensive) resource: frequencies in the radio spectrum.

Like with all <u>scarcely available resources</u>, the cost of frequency-use provides the need for economic-use of the available frequencies.

Reuse of frequencies within a wireless communication network can offer considerable economies.

…

The Problem (8)

However, reuse of frequencies may also lead to a loss of quality of communication links.

Namely, the use of (almost) the same frequency for multiple wireless connections can cause an interference between the signals that is unacceptable.

A solution to the frequency assignment problem balances the economies of reuse of frequencies and the loss of quality in the network.

Quantification of the different aspects results in a mathematical optimization problem.

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The Problem (9)

In general, Frequency Assignment Problems (FAPs) have two basic aspects:

1. a set of wireless communication connections must be assigned frequencies such that, for every connection, data transmission between the transmitter and receiver is possible. The frequencies should be selected from a given set that may depend on the location.
2. The frequencies assigned to two connections may incur interference resulting in a loss of quality of the signal.

But, what interference is?

The Problem (10)

Two conditions must be fulfilled in order to have interference of two signals:

a) The two frequencies must be close on the electromagnetic band (Doppler effects) or (close to) harmonics of one another.

   The latter effect is limited, since the frequency bands from which we can choose are usually so small that they do not contain harmonics.

b) The connections must be geographically close to each other.

The Problem (11)

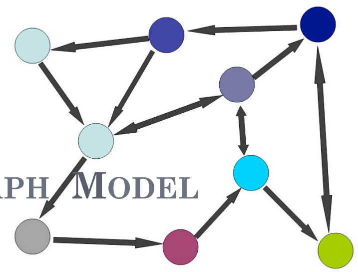
Both aspects are modeled in many different ways in the literature.

Hence: various models.

They differ in the types of constraints and in the objectives to be optimized.

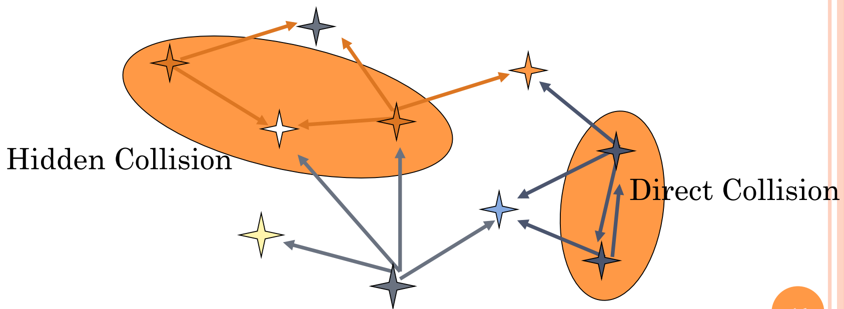
Here we describe a simplified model.

The Graph Model



The Model (1)

In our model:



THE MODEL (2)

Interference due to re-use of the same channel at "close" or "very close" sites.

Contributions also from sites using only close channels, since in practice transceivers do not operate exclusively within the frequencies of their assigned channels.

Direct Collisions: stations positioned in close locations receive channels at least h apart

Hidden Collisions: channels for stations positioned in very close locations are at least k apart

⬇

L(h,k)-Labeling Problem

L(*h*,*k*)-LABELING (1)

Interference Graph

- One node per station
- One edge between two stations if the may communicate (and hence interfere)
- Labels interpreted as channels assigned to the nodes.

f: node coloring function s.t.

- $\forall\ u, v \in V\ |f(u) - f(v)| \geq \boldsymbol{h}$ if $(u,v) \in E$
- $\forall\ u, v \in V\ |f(u) - f(v)| \geq \boldsymbol{k}$ if $\exists\ w \in V$ s.t. $(u,w) \in E$ and $(w,v) \in E$

Objective: minimizing the *bandwidth* $\sigma_{h,k}$

Minimum bandwidth: $\lambda_{h,k}$

L(*h*,*k*)-LABELING (2)

Obs. The condition**:**

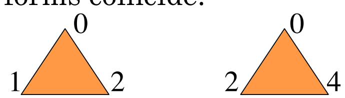
- $\forall\ u, v \in V\ |f(u) - f(v)| \geq \boldsymbol{k}$ if $\exists\ w \in V$ s.t. $(u,w) \in E$ and $(w,v) \in E$

is often written as:

- $\forall\ u, v \in V\ |f(u) - f(v)| \geq \boldsymbol{k}$ if $\text{dist}(u,v)=2$

The first form works both when $\boldsymbol{h} \geq \boldsymbol{k}$ and when $\boldsymbol{h} < \boldsymbol{k}$.
It allows a triangle to be labeled with colors at mutual distance at least $\max\{\boldsymbol{h},\boldsymbol{k}\}$, even if its nodes are at distance 1.
When $\boldsymbol{h} \geq \boldsymbol{k}$ the two forms coincide.

L(1,2)



L(*h*,*k*)-LABELING (3)

Usually, the minimum used color is *0*.
So, an $L(h,k)$-labeling having span $\sigma_{h,k}(G)$ uses $\sigma_{h,k}(G)+1$ different colors.
This is slightly counter-intuitive, but it is used for historical reasons.

The problem has been introduced in the '90s with $h=2$ and $k=1$ in relation with a frequency assignment problem

[Griggs e Yeh '92, Robertson '91]

This problem was already known in combinatorics in the case $h=1$ and $k=1$ (coloring the square of a graph)

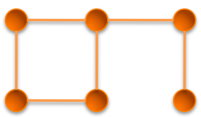
[Wegner '77]

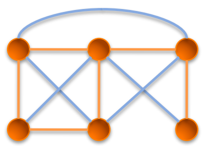
A Parenthesis on the L(*1,1*)-Labeling (1)

When $h=1$ and $k=1$ the problem is equivalent to the classical vertex coloring of the square of a graph.

Given a graph $G=(V, E)$, its square G^2 is defined as a graph having node set equal to V, and an edge between u and v is in G^2 iff:

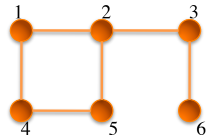
- either (u,v) is in E
- or u and v are connected by a length 2 path in G

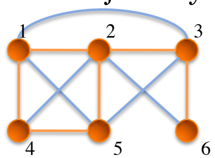




A Parenthesis on the L(*1,1*)-Labeling (2)

If the graph is stored in an adjacency matrix:





Compute $A(G)^2$:

$(a_{ij})^2=\sum_{k=1\ldots n} a_{ik}\, a_{kj}$

$$A(G)=\begin{matrix}0&1&0&1&0&0\\1&0&1&0&1&0\\0&1&0&0&0&1\\1&0&0&0&1&0\\0&1&0&1&0&0\\0&0&1&0&0&0\end{matrix} \qquad A(G^2)=\begin{matrix}0&1&1&1&1&0\\1&0&1&1&1&1\\1&1&0&0&1&1\\1&1&0&0&1&0\\1&1&1&1&0&0\\0&1&1&0&0&0\end{matrix} \qquad A(G)^2=\begin{matrix}2&0&1&0&2&0\\0&3&0&2&0&1\\1&0&2&0&1&0\\0&2&0&2&0&0\\2&0&1&0&2&0\\0&1&0&0&0&1\end{matrix}$$

$(a_{ij})^2=x$ iff there are x 2-length paths between i and j.
To store together the knowledge about 1-and 2-length paths:
$A(G)^2+A(G) \to A(G^2)$

L(*h,k*)-Labeling (4)

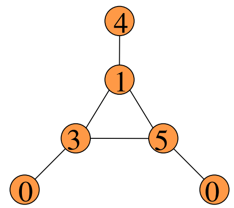
After its definition, the $L(h,k)$-labeling problem has been used to model several problems:

- a kind of integer 'control code' assignment in packet radio networks to avoid hidden collisions ($L(0,1)$-labeling problem)
- channel assignment in optical cluster-based networks ($L(0,1)$- or $L(1,1)$-labeling depending on the fact that the clusters can contain one ore more nodes)
- more in general, channel assignment problems, with a channel defined as a frequency, a time slot, a control code, etc.

$L(h,k)$-labeling has been studied following many different approaches: graph theory and combinatorics, simulated annealing, genetic algorithms, tabu search, neural networks, …

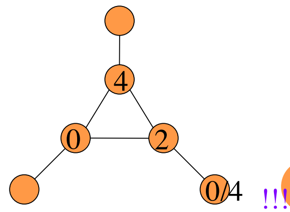
L(*h,k*)-Labeling (5)

Example: $L(2,1)$-labeling of:



$\lambda_{2,1} \leq 5$

$\lambda_{2,1}=5$: by contradiction



L(h,k)-LABELING (6)

Lemma: $\lambda_{dh,dk}=d\,\lambda_{h,k}$

Proof. Divided into two parts: $\lambda_{dh,dk}\geq d\,\lambda_{h,k}$ and $\lambda_{dh,dk}\leq d\,\lambda_{h,k}$.

1. $\lambda_{dh,dk}\geq d\,\lambda_{h,k}$

Let f an $L(dh, dk)$-labeling. Define $f'=f/d$.
f' is an $L(h,k)$-labeling and $\lambda_{dh,dk}/d = \sigma_{h,k}(f') \geq \lambda_{h,k}$.

2. $\lambda_{dh,dk}\leq d\,\lambda_{h,k}$

Similarly, let f an $L(h,k)$-labeling. Define $f'=fd$.
f' is an $L(dh, dk)$-labeling and $\lambda_{dh,dk} \leq \sigma_{dh,dk}(f') = d\,\lambda_{h,k}$. ■

It follows that we can restrict to use values of h and k mutually prime.

L(h,k)-LABELING (7)

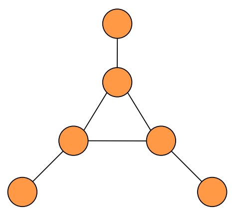
PROBLEM: What if $f'=f/d$ does not produce integer values?

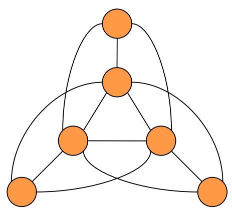
Lemma: Let $x, y \geq 0$, $d>0$ and k in Z^+.
If $|x-y|\geq kd$, then $|x'-y'|\geq kd$,
where $x' = \lfloor x/d \rfloor d$ and $y' = \lfloor y/d \rfloor d$

It follows we can restrict to use values of h and k integer and mutually prime.

L(h,k)-LABELING (8)

- The case $k=0$, for any h, is not usually considered as an $L(h,k)$-labeling problem, as it coincides with the classical vertex coloring
- The case $h=k$ is very studied in the literatue as the vertex coloring of the square of a graph.
- The case $h=2k$ is the most studied.





NP-COMPLETENESS RESULTS (1)

The decisional version of the problem is NP-complete, even for small values of h and k:

$L(0,1)$-labeling of planar graphs

[Bertossi, Bonuccelli '95]

$L(1,1)$-labeling of general, planar, bounded degree and e unit-disk graphs

[McCormick '83], [Ramanathan, Loyd '92], [Ramanathan '93], [Sen, Huson '97]

NP-completeness Results (2)

Th. The $L(2,1)$-labeling problem on diam. 2 graphs is NP-complete [Griggs, Yeh '92]

Proof. Consider the following special form of the decisional problem:

DL. Instance: $G=(V,E)$ diam. 2 graph

Question: $\lambda_{2,1}(G) \leq |V|$?

it never maps distinct elements of its domain to the same element of its codomain

IDL. Instance: $G=(V,E)$

Question: Does exist an f injective s.t.

$|f(x)-f(y)| \geq 2$ if $(x,y) \in E$

and its codomain is $\{0, \ldots, |V|-1\}$?

NP-completeness Results (3)

(proof of NP-completeness cntd)

Finding a solution for IDL means finding an hamiltonian path in G^C:

Since f is injective, f^{-1} is defined.

Give an order to nodes:

$v_i=f^{-1}(i),\ 0 \leq i \leq |V|-1$

Observe that, since v_i and v_{i+1} cannot be adjacent in G, they are adjacent in G^C, hence

$v_0, v_1, \ldots, v_{|V|-1}$ is a hamiltonian path.

NP-completeness Results (4)

(proof of NP-completeness cntd)

Even the reverse holds:

Given a hamiltonian path in G^C

$v_0, v_1, \ldots, v_{|V|-1}$ define f such that $f(v_i)=i$,

$0 \leq i \leq |V|-1$.

f is trivially injective; furthermore, given an edge $\{x,y\}$ of G, $x=v_i$ and $y=v_j$, it must hold:

$|f(x)-f(y)| \geq 2$ because x and y are not adjacent in G^C.

It follows that the two problems are equivalent.

NP-completeness Results (5)

(proof of NP-completeness cntd)

The following problem:

HP. Instance: $G=(V,E)$

Question: Does G have a hamiltonian path?

is NP-complete, so even IDL is NP-complete.

Instance: $G=(V,E)$ diam. 2 graph

Question: $\lambda_{2,1}(G) \leq |V|$?

DL is in NP:

We can verify in polynomial time that G has diameter 2, whether a labeling f is a feasible $L(2,1)$-labeling, and whether $\lambda_{2,1}(G) \leq ||f(G)|| \leq |V|$.

NP-completeness Results (6)

(proof of NP-completeness cntd)

Transformation from IDL to DL to prove that DL is NP-complete:

Given an instance of IDL G, construct G':

- $V'=V \cup \{x\}$
- $E'=E \cup \{\{x,a\}$ for each a in $V\}$

So $|V'|=|V|+1$ and G' has diameter 2

NP-completeness Results (7)

(proof of NP-completeness cntd)

We prove that from a solution for DL it is possible to deduce a solution for IDL, i.e. there is an injection f s.t.

$|f(x)-f(y)|\geq 2$ for every $(x,y) \in E$ iff $\lambda_{2,1}(G')\leq|V'|$.

- => If there exists an injection f defined on V that satisfies the condition above, define $g(v)=f(v)$ for all $v\in V$ and $g(x)=|V|+1=|V'|$.

Easily g is an $L(2,1)$-labeling for G' and $\lambda_{2,1}(G') \leq ||g(G')|| \leq |V'|$

NP-completeness Results (8)

(proof of NP-completeness cntd)

- <= Conversely, suppose that $\lambda_{2,1}(G')\leq|V'|$, i.e. there exists a feasible $L(2,1)$-labeling g s.t. $||g(V')||\leq|V'|=|V|+1$.

Observe that G' of diam. 2 implies that

$g(a)\neq g(b)$ for each $a\neq b$

- Suppose $g(x)\neq|V|+1$ and $\neq 0$. By the property of $L(2,1)$-labeling, there is no v in V such that $g(v)=g(x)-1$ or $g(x)+1$. So we need $|V|+3$ labels for V' i.e. $\lambda_{2,1}(G')\geq|V'|+1$: a contardiction.

NP-completeness Results (9)

(proof of NP-completeness cntd)

- So $g(x)$ is either 0 or $|V|+1$.
  - If $g(x)=|V|+1$ => $f(v)=g(v)$ OK
  - If $g(x)=0$ => $f(v)=g(v)-2$ OK

In any case, there exists f injective s.t. its codomain is $\{0, \ldots, |V|-1\}$.

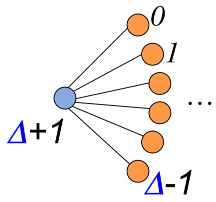
The NP-compleness of DL follows. ■

Literature in different directions:

- Lower and upper bounds for $\lambda_{h,k}$
- Limitation to special graph classes:
  - Exact labelings
  - Approximate labelings

Lower Bounds (1)

- $\lambda_{2,1} \geq \Delta+1 = (\Delta-1)1+2$
- $\lambda_{h,k} \geq (\Delta-1)k+h$ for $h \geq k$

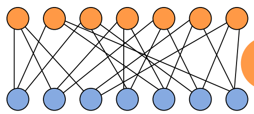


Lower Bounds (2)

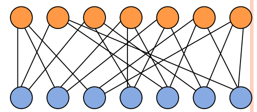
- $\exists$ G s.t. $\lambda_{2,1}(G) \geq \Delta^2 - \Delta$ [Griggs, Yeh '92]

Incidence graph of a projective plane $\pi(n)$ of order n, $G=(U \cup V, E)$ s.t.

- $|U|=|V|=n^2+n+1$
- $u \in U$ corresonds to a point P_u of $\pi(n)$
- $v \in V$ corresonds to a line l_v of $\pi(n)$
- $E=\{(u,v)$ s.t. $P_u \in l_v\}$



Lower Bounds (3)



- G is regular and $\Delta=n+1$
- For each pair of nodes in U (or in V), their distance is 2,
- $\forall$ $u,v \in U$ ($\in V$), $|Adj(u) \cap Adj(v)|=1$

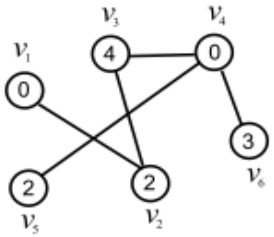
$\Rightarrow$ $\lambda_{2,1}(G) \geq |U|-1=|V|-1=\Delta^2 - \Delta$

Upper bounds (1)

Greedy Algorithm:

Given a graph G with nodes $v_1, v_2, \dots, v_n$,

label its nodes in order assigning to v_i the smallest color not conflicting with the labels of its neighborhood (dist. 1 and 2)



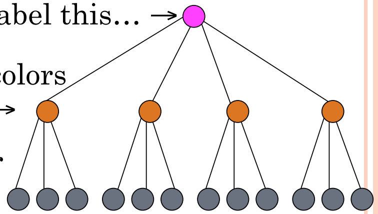
Upper bounds (2)

- Th. $\lambda_{2,1}(G) \le \Delta^2+2\Delta$ [Griggs, Yeh '92]
- Proof.

In order to label this… →

…we have to eliminate at most 3 colors for each one of these… →

…and at most one color for each one of these… →

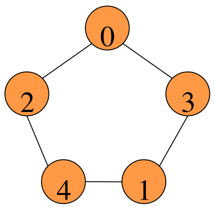


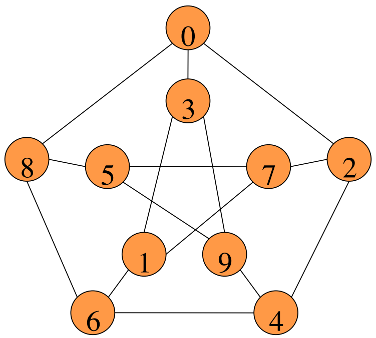
We can label all the graph with at most $1+3\Delta+(\Delta-1)\Delta$ colors.

Upper bounds (3)

Conjecture: $\lambda_{2,1}(G) \le \Delta^2$ [Griggs, Yeh '92]

This upper bound is tight: some graphs with degree Δ, diameter 2 and Δ^2+1 nodes have λ at leats Δ^2.





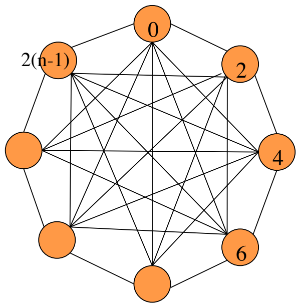
Upper bounds (4)

Conjecture: $\lambda_{2,1}(G) \le \Delta^2$ [Griggs, Yeh '92]

- $\lambda_{2,1}(G) \le \Delta^2+2\Delta-4$ [Jonas '93]
- $\lambda_{2,1}(G) \le \Delta^2+\Delta$ [Chang, Kuo '96]
- $\lambda_{2,1}(G) \le \Delta^2+\Delta-1$ [Kral, Skrekovski '03]
- $\lambda_{2,1}(G) \le \Delta^2+\Delta-2$ [Goncalves '05]
- $\lambda_{2,1}(G) \le \Delta^2$ for sufficiently large values of Δ [Havet, Reed and Sereni '08]

EXACT RESULTS: CLIQUES K_n

- $\lambda_{2,1}(K_n)=2(n-1)$
- All nodes are pairwise adjacent



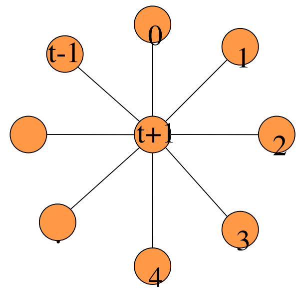
EXACT RESULTS: STARS $K_{1,t}$

- $\lambda_{2,1}(K_{1,t})=t+1$

Proof.

- $\lambda_{2,1}(K_{1,t})\leq t+1$ easy
- $\lambda_{2,1}(K_{1,t})\geq t+1$

  by contradiction



EXACT RESULTS: TREES T_n (1)

- $\lambda_{2,1}(T_n)=\Delta+1$ or $\Delta+2$

Proof.

- $\lambda_{2,1}(T_n)\geq\Delta+1$ because T_n contains a $K_{1,\Delta}$
- $\lambda_{2,1}(T_n)\leq\Delta+2$

  first-fit (greedy) labeling:

  Order the nodes of T_n: $T_{n-1}=T_n\text{-}\{v_n\}$ where v_n is a leaf.

  In general $T_i=T_{i+1}\text{-}\{v_{i+1}\}$

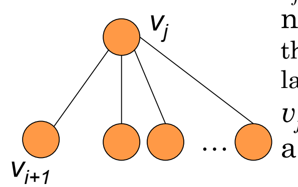
  Label v_1 with 0.

  Label v_i with the first available color.

EXACT RESULTS: TREES T_n (2)

(proof: either $\lambda_{2,1}(T_n)=\Delta+1$ or $\Delta+2$ – cntd)

Assume we have already labeled all nodes from v_1 to v_i and we are going to label v_{i+1}:



v_j parent of v_{i+1}

necessarily $j\leq i+1$ (the nodes that are closer to the leaves have larger numebring)

v_j has at most $\Delta\text{-}1$ further adjacent nodes

At most 3 colors are forbidden due to v_j

At most $\Delta\text{-}1$ colors are forbidden due to the nodes that are adjacent to v_j

If we have at least $(\Delta\text{-}1)+3+1$ colors, we are always able to label v_{i+1} i.e. $\lambda_{2,1}(T_n)\leq\Delta+2$. ■

EXACT RESULTS: TREES T_n (3)

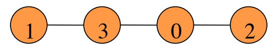
- This proof has been proposed by Griggs e Yeh ['92], who have also conjectured that it is NP-complete to decide whether the correct value is $\Delta+1$ or $\Delta+2$.
- Chang e Kuo ['96] have disproved this conjecture by providing a polynomial algorithm based on the dynamic programming technique and having time complexity $O(\Delta^{4.5}\, n)$.
- Many authors have proposed many other algorithms aiming at improving the time complexity.
- Finally, Hasunama, Ishi, Ono, Uno ['08] have proposed a linear algorithm.

EXACT RESULTS: PATHS P_n

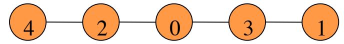
- $\lambda_{2,1}(P_2)=2$
- $\lambda_{2,1}(P_3)=3$

From the results for the stars

- $\lambda_{2,1}(P_4)=3$

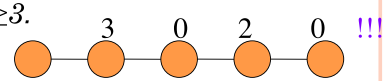


- $\lambda_{2,1}(P_n)=4$ if $n\geq 5$



To prove that $\lambda_{2,1}(P_5)\leq 4$:

P_5 includes a P_4 so $\lambda_{2,1}(P_5)\geq 3$.
By contradiction $\lambda_{2,1}(P_5)=3$

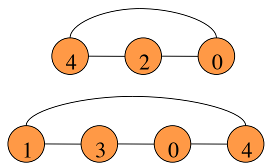


If $n\geq 5$ the result follows from the previous one and from the result for trees.

EXACT RESULTS: CYCLES C_n (1)

- $\lambda_{2,1}(C_n)=4$

If $n\leq 4$: case by case:



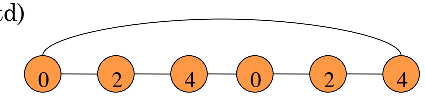
If $n\geq 5$: C_n contains P_n so $\lambda_{2,1}(C_n)\geq 4$.
It also holds $\lambda_{2,1}(C_n)\leq 4$:

3 cases: …

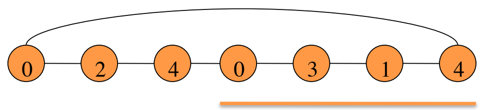
EXACT RESULTS: CYCLES C_n (2)

(proof: $\lambda_{2,1}(C_n)=4$ – cntd)

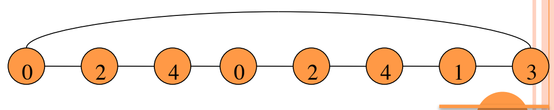
1. $n=0 \pmod 3$



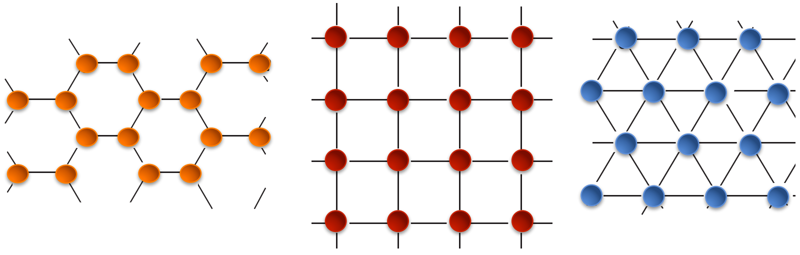
2. $n=1 \pmod 3$



3. $n=2 \pmod 3$

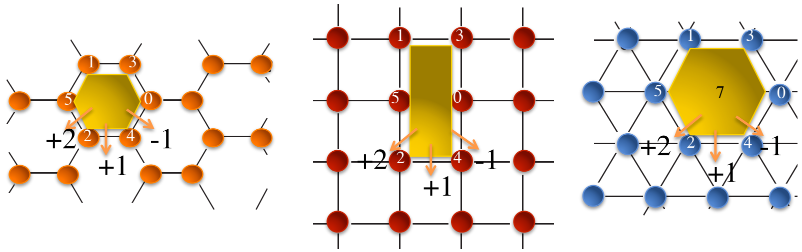


Exact Results: Grids (1)

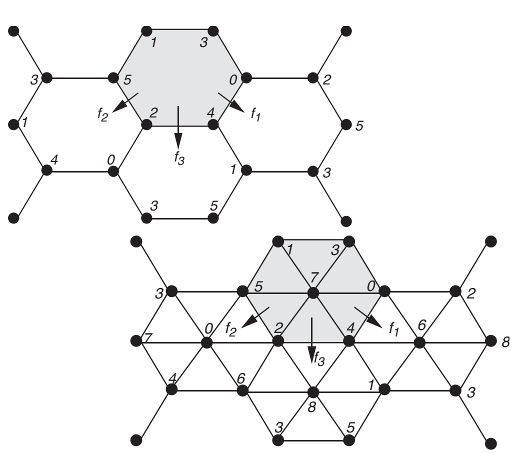


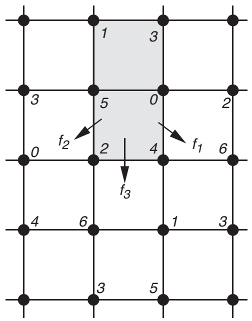
Exact Results: Grids (2)

$\lambda_{2,1}(\Delta)=\Delta+2$



Exact Results: Grids (3)



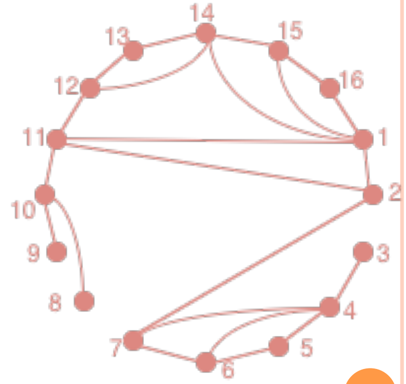


Approximate Results: Outerplanar Graphs(1)

- $\lambda_{2,1}(G) \leq 2\Delta+4$ because G has treewidth 2
- Jonas ['93]: $\lambda_{2,1}(G) \leq 2\Delta+2$
- Bodlaender et al. ['04]: $\lambda_{2,1}(G) \leq \Delta+8$ but they conjecture that $\lambda_{2,1}(G) \leq \Delta+2$ -> possible students' lesson
- C.& Petreschi ['04] $\Delta+1 \leq \lambda_{2,1}(G) \leq \Delta+2$ and they conjecture that this algorithm gives the optimum value.

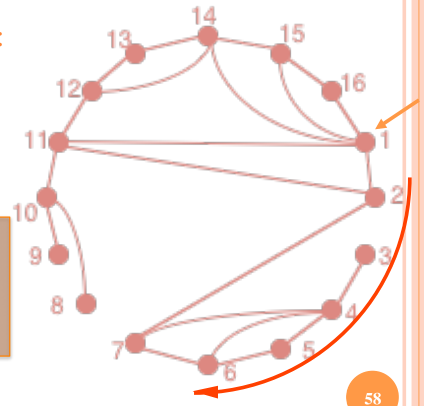
APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(2)

Def. A graph is said to be outerplanar if it can be represented as a plane graph so that each node lies on the border of the external face

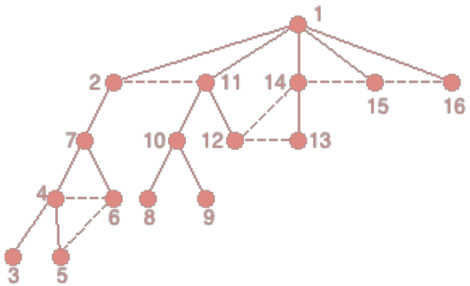


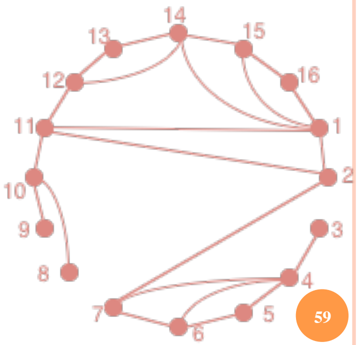
APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(3)

- Ordered Breadth First Tree:
- Choose a node r
- Induce a total order on the nodes on the external face
- Run a BFS from r so that nodes coming before in the orderong are visited before than the others.



APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(4)





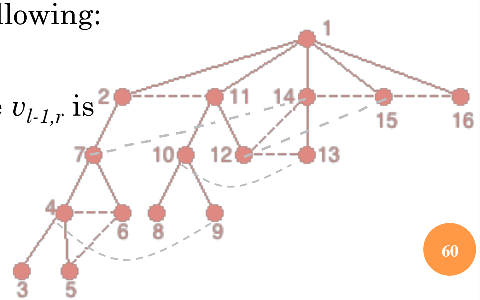
APPROXIMATE RESULTS: OUTERPLANAR GRAPHS(5)

Th. (well known)

G=(V, E); BFT $T=(V, E')$

Each non tree edge $(v_{l,h}, v_{l',k})$ satisfies one of the following:

- $l'=l$
- $l'=l-1$ and $r<k$, where $v_{l-1,r}$ is the parent of $v_{l,h}$.

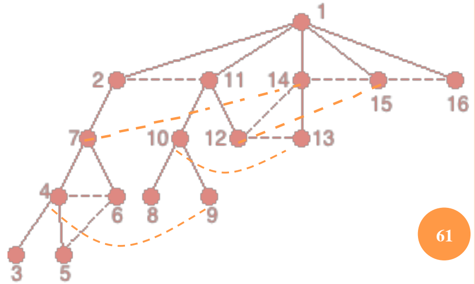


Approximate Results: Outerplanar Graphs(6)

Th. G=(V, E); OBFT $T=(V, E')$:

- If $(v_{l,h}, v_{l,k})$, $h<k$, then $k=h+1$
- If $v_{l,h}$ is a child of $v_{l-1,i}$ and $(v_{l,h}, v_{l-1,k})$ is a non tree edge, $i<k$, then $k=i+1$ and $v_{l,h}$ is the rightmost of its siblings

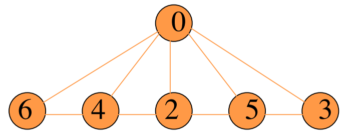
(orange edges are not admissible)

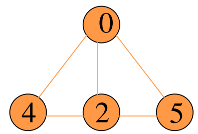


Approximate Results: Outerplanar Graphs(7)

Lemma. If $\Delta \geq 4$, $\Delta+2$ colors are necessary.

Proof.



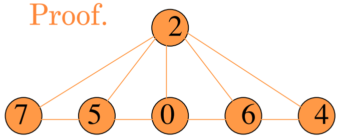


$\boxed{0\ \ 1}$ 2 3 4 5 6

Approximate Results: Outerplanar Graphs(8)

- Lemma. If the root of the tree has an already assigned color, then $\Delta+3$ colors are necessary.

Proof.



0 $\boxed{1\ \ 2\ \ 3}$ 4 5 6 7

General technique

Approximate Results: Outerplanar Graphs(9)

- INPUT: G outerplanar with max degree Δ
- OUTPUT: a feasible $L(2,1)$-labeling for G
- Consider a node v with max degree and run an OBFS starting from v
- $Label(v) \leftarrow 0$
- Label the first layer according to the previous Lemma
- For each layer $l \geq 2$, top-down, from left to right

  repeat

  Label the children of $v_{l,k}$ according to the previous Lemma, eliminating from the palette the forbidden colors

Approximate Results: Outerplanar Graphs(10)

- INPUT: G outerplanar with max degree Δ
- OUTPUT: a feasible $L(2,1)$-labeling for G

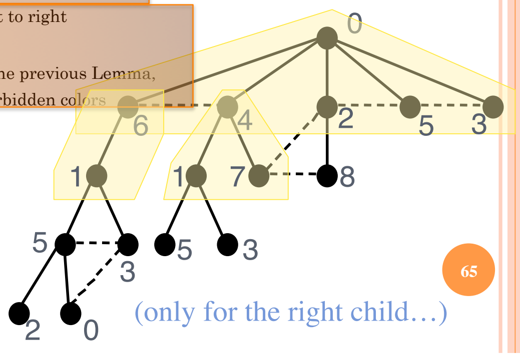
An example:

- Consider a node v with max degree and run an OBFS starting from v
- $Label(v) \leftarrow 0$
- Label the first layer according to the previous Lemma
- For each layer $l \geq 2$, top-down, from left to right
  repeat
  Label the children of $v_{l,k}$ according to the previous Lemma, eliminating from the palette the forbidden colors

0 1 2 3 4 5 6

0 1 2 3 4 5 6

0 1 2 3 4 5 6 7



(only for the right child…)

Approximate Results: Outerplanar Graphs(11)

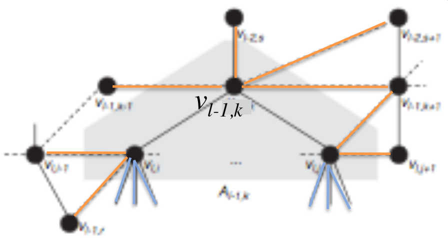
Correctness and Bounds

- Best previously known results: $\lambda \leq \Delta+8$ for each Δ [BKTL'00]
- Conjecture [BKTL00]:$\lambda \leq \Delta+2$ for each Δ
- Th. The provided algorithm correctly $L(2,1)$-labels each outerplanar graph with max degree $\Delta \geq 8$ with $\lambda \leq \Delta+2$ in linear time; otherwise, at most 11 colors are anyway necessary.
- Proof. By induction, considering the edges coming out from the subgraph induced by any node and its children…

Approximate Results: Outerplanar Graphs(12)

- a. one tree-edge to the father of $v_{l-1,k}$
- b. at most three non-tree edges connecting $v_{l-1,k}$ with some nodes either at the same layer or at the previous layer
- c. at most two non-tree edges from the leftmost sibling
- d. at most two non-tree edges from the rightmost sibling

The other outgoing edges do not contribute to the labeling during this step.

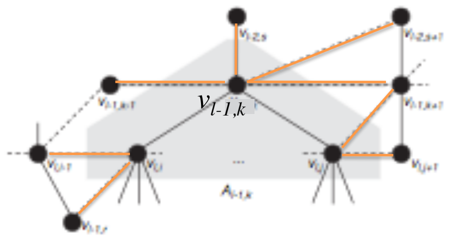


Approximate Results: Outerplanar Graphs(13)

(Proof sketch) By inductive hypothesis at most $\Delta+3$ colors have been used. We prove that they are sufficient to label the children of $v_{l-1,k}$.

We cannot use:

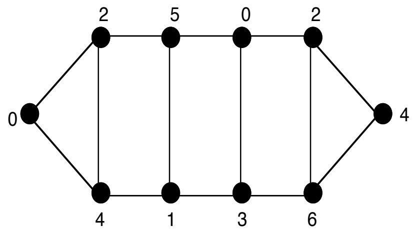
- At most 3 colors (for $v_{l-1,k}$) ➔Δ
- At most 1 color (edge a) ➔$\Delta-1$
- At most x colors (edges b), $0 \leq x \leq 3$ ➔$\Delta-1-x$ for the $\Delta-1-x$ children of $v_{l-1,k}$
- The c edges give conditions only on 1 or 2 nodes, hence it is possible to arrange
- Analogously for the d edges ■



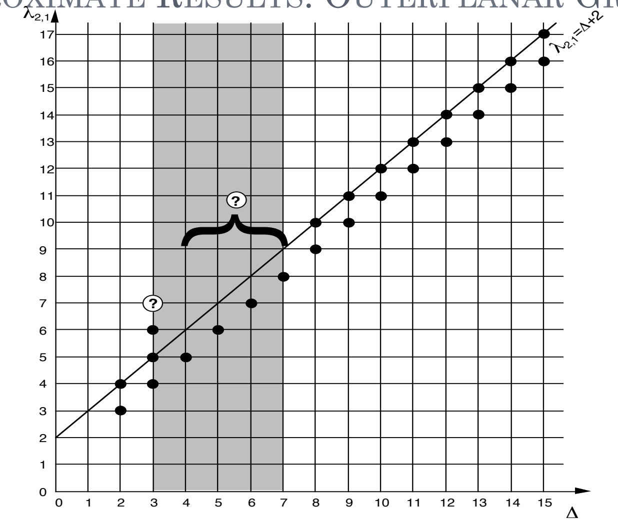
Approximate Results: Outerplanar Graphs(14)

The special case $\Delta=3$

- There exists an infinite class of outerplanar graphs having $\Delta=3$ requiring $\lambda=\Delta+3$
- It is possible to provide a labeling algorithm for these graphs using $\lambda\leq\Delta+5$



Approximate Results: Outerplanar Graphs(15)



Variations of the Problem

Variations of the Problem (1)

oriented L(2,1)-Labeling

- An oriented L(2,1)-labeling of a directed graph G is a function assigning colors from 0, … ,λ to the nodes of G so that nodes at distance 2 in the graph take different colors and adjacent nodes take colors at distance 2.
- Oriented L(2,1)-labeling problem

  minimizing λ
- Note. The minimum value of λ can be very different from the value of the same parameter in the undirected case. Example: trees…

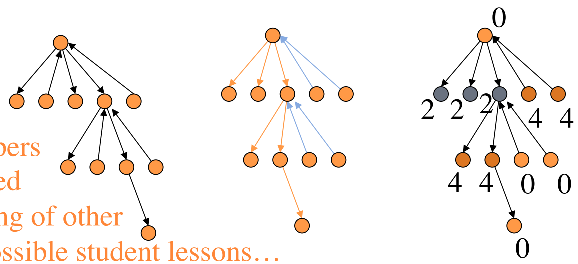
Variations of the Problem (2)

Oriented L(2,1)-Labeling

- Reminder: In undirected trees, $\Delta+1 \leq \lambda \leq \Delta+2$, and the exact value is linearly decidible

  [Chang & Kuo '96, Hasunama et al. 2008]

- In directed trees, $\lambda \leq 4$ [Chang & Liaw '03]



Available papers on the oriented L(2,1)-labeling of other graphs for possible student lessons…

Variations of the Problem (3)

$L(h_1, \ldots, h_k)$-Labeling

With the aim of making the model more realistic:

- An $L(h_1, \ldots, h_k)$-labeling of a graph G is a function assigning integer values to the nodes of the graph such that:

  $|l(u)-l(v)|\geq h_i$ if u and v are at distance i in the graph, $1\leq i\leq k$.
- $L(h_1, \ldots, h_k)$-labeling problem: minimizing λ
- Particularly interesting: $L(2,1,1)$ and $L(\delta, 1, \ldots, 1)$.
- Even these special cases are NP-hard on general graphs, so special classes of graphs are handled.

Variations of the Problem (4)

Backbone Coloring

If the topology has a backbone, where the transmitting power is higher wrt the rest of the network:

- A *Backbone coloring* of a graph G wrt a graph H is a function assigning integer values to the nodes of the graph such that:

  $|l(u)-l(v)|\geq 2$ if (u,v) is an edge of H and

  $|l(u)-l(v)|\geq 1$ if (u,v) is an edge og G-H.
- *Backbone coloring problem*:

  minimizing λ

Available papers on this coloring for student lessons…

Variations of the Problem (5)

n-multiple L(h,k)-Labeling

In practice, each transmitting station is able to handle more than one channel, so a set of channels is assigned to it.

- Given two set of integer values I and J, we define

  $dist(I,J)=min\{|i-j|: i \text{ in } I \text{ and } j \text{ in } J\}$

Example:

$I=\{0,1,2\}$; $J=\{4,5,6\}$; $dist\{I,J\}=2$.

Variations of the Problem (6)
n-multiple L(h,k)-Labeling

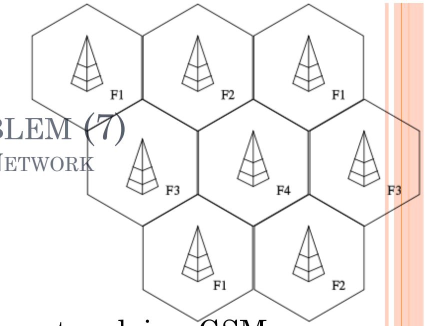
- An *n-multiple L(h,k)-labeling* of a graph G is a function assigning n integer values to each node of the graph so that:

  $dist(l(u),l(v)) \geq h$ if (u,v) is an edge of G and

  $dist(l(u),l(v)) \geq k$ if u and v are at dist. 2 in G.
- *n-multiple L(h,k)-labeling problem*:

  minimizing λ, given n.

Variations of the Problem (7)
Frequency Assignment in a GSM Network



In the special case in which the network is a GSM:

- The network is a cellular network with hexagonal cells.
- Each cell has its own station connecting the fixed network devices with the mobile devices that are at moment inside the cell.
- Mobile phones connect to the GSM network trying to communicate with the station associated to the cell where they lie.

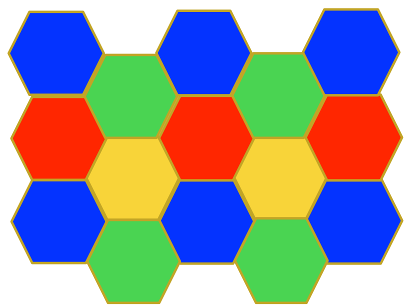
Variations of the Problem (8)
Frequency Assignment in a GSM Network

In a GSM network, the cells need to use different frequences, in order not to interfere.

- Coloring map problem: given a plane map, the problem consists in coloring each region in such a way that adjacent regions take different colors and that the min number of colors is used.
- Four Color Theorem: It is always possible to color a map using at most 4 colors.

Variations of the Problem (9)
Frequency Assignment in a GSM Network

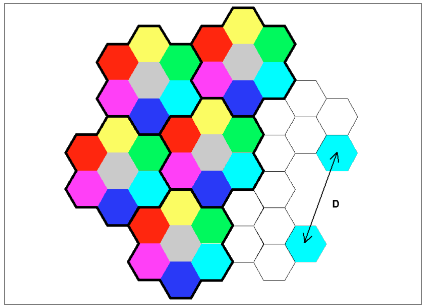
- It follows that 4 different frequencies are sufficient for an arbitrary GSM network:



VARIATIONS OF THE PROBLEM (10)
FREQUENCY ASSIGNMENT IN A GSM NETWORK

- In fact, more sofisticated variations of this problem lead to 7 colors -> L(h,k)-labeling again

More in detail…



VARIATIONS OF THE PROBLEM (11)
FREQUENCY ASSIGNMENT IN A GSM NETWORK

- In wireless communication networks of the 1st and 2nd generation, the concept of cellular channel allocation and spatial frequency reuse were the key ideas that have driven the initial success of mobile telephony.
- For example, a seven color labeling of a hexagonal grid was on the basis of the AMPS (American Mobile Phone System).
- The same scheme existed for GSM.
- …

VARIATIONS OF THE PROBLEM (12)
FREQUENCY ASSIGNMENT IN A GSM NETWORK

- In the 3rd generation of mobile systems, the introduction of CDMA (Code Division Multiple Access) has enabled the reuse of the whole frequency band in each cell: instead of dividing the signal space in time or frequency, a code of pseudorandom sequence is used to differentiate the signal from each transmitter.
- In this context, the labeling schemes were of much reduced importance.
- …

VARIATIONS OF THE PROBLEM (13)
FREQUENCY ASSIGNMENT IN A GSM NETWORK

- The 4th generation mobile standards mainly use Orthogonal Frequency Division Multiple Access schemes.
- These schemes divide the signal space in time slots and orthogonal frequencies.
- At the middle of a cell, all slots of time and frequencies are allocated to users. At the edge of a cell, only part of the band is used and a three color scheme is used.
- Even this model can be reduced to a labeling scheme. For further details:

[Archetti, Bianchessi, Hertz, Colombet, Gagnon '13]

A Parenthesis on the 4 Color Problem (1)

- Given a map, it can be naturally considered a planar graph G.
- Given G, let G^* its dual graph:
  - Put a node of G^* in each region of G
  - Connect two nodes of G^* iff the corresponding regions (faces) are adjacent (i.e. share an edge in G)
- A vertex coloring of G^* corresponds to a map coloring of G.

A Parenthesis on the 4 Color Problem (2)

- In fact, cartographers have always known that 4 colors were enough for each kind of map, but in 1852 Francis Guthrie wondered whether this fact could be proved.
- After more than 100 years, and many (wrong) announcements, Appel and Haken proved the 4 Color Theorem in 1976.
- The complete proof is computer assisted because it exhaustively examines more than 1700 configurations.
- More recently, Robertson, Sanders, Seymour, and Thomas wrote a new proof, needing to examine "only" 633 configurations.

A Parenthesis on the 4 Color Problem (3)

There are some interesting results for other numbers of colors:

- **2-coloring**.

  Polynomially solvable:
  - Assign a color to a region.
  - Assign the other color to its neighbor regions.
  - Assign the first color to its neighbor regions.
  - Continue until the regions have been all colored or there is a color conflict. In this latter case the map is not 2-colorable.

A Parenthesis on the 4 Color Problem (4)

- **3-coloring**
  - NP-hard, hence no algorithms to decide whether a map is 3-colorable or not.
  - Method: exhaustively try all the color combinations for the regions.
  - Inapplicable: for N regions, there are 3^N possibilities. (if $N=48$ the combinations are about 8×10^{22})
  - …

A Parenthesis on the 4 Color Problem (5)

- **3-coloring (cntd)**
  - There are some techniques in order to simplify the map before coloring it (for example, if a region has only 2 neighbor regions, it can be eliminated from the map: when it is re-inserted, it will be colored with the third color) but the worst case time complexity is the same.

A Parenthesis on the 4 Color Problem (6)

- **4-coloring**
  - The proof of the 4 color theorem is constructive, and so it shows how to find a feasible coloring, but the number of cases is too high to be useful in practice.
  - There are some transformations, similar to those used for the 3-coloring, but they do not eliminate the need of exhaustively try all the possibilities.

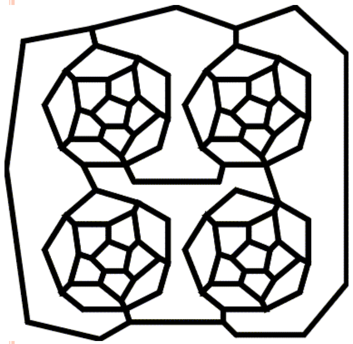
A Parenthesis on the 4 Color Problem (7)

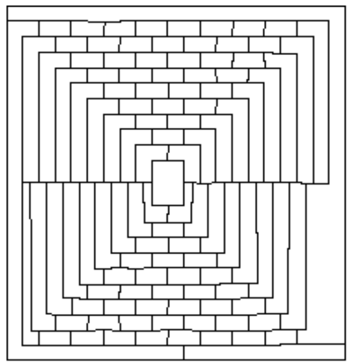
- **5-coloring**
  - It is relatively easy to color a map using 5 colors. There is an algorithm that first simplifies the map eliminating all the regions and then re-insert them assigning the correct color.

A Parenthesis on the 4 Color Problem (8)

We conclude with a puzzle:

Try to 4-color these 2 maps…





In 1975 Martin Gardner claimed he could prove that this map was not 4-colorable (April fool)

A Parenthesis on the 4 Color Problem (9)

Solutions