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Improved approximation algorithms for minimum AND-circuits problem via *k*-Set Cover

NETWORK ALGORITHMS PRESENTATION BY MATTEO PRATA - 13/11/2018

Introduction to the problem

"Improved approximation algorithms for minimum AND-circuits problem via k-set cover"

By Hiroki Morizumi (2010)

- Approximation algorithms: **Cover-μ** and **Greedy-μ**
- **Minimum AND-circuit**, a circuit minimization problem
- Reduction to k-Set Cover
- Approximation ratio of at most 1.199 (outperforms 1.278 by J. Arpe and B. Manthey)

Preliminaries 1/2

- A **Boolean monomial** over a set of boolean variables $X = \{x_1, \dots, x_n\}$ is an AND-product of variables of a subset of X

$$M = x_{i_1} \wedge \dots \wedge x_{i_d} = \{x_{i_1}, \dots, x_{i_d}\}$$

- We denote by $\binom{M}{2}$ the set $\{S \subseteq X \mid |S| = 2 \wedge S \subseteq M\}$

- Given a set of monomials $\mathcal{M} = \{M_1, \dots, M_k\}$ the ***multiplicity*** of a set $S \subseteq X$ is the number of occurrences of S as a sub-monomial of a monomial i.e.

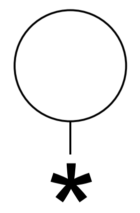
$$\mathrm{mult}_{\mathcal{M}}(S) = |\{M \in \mathcal{M} \mid S \subseteq M\}|$$

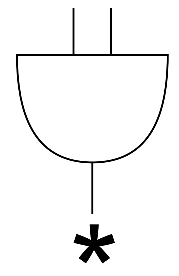
- The ***maximum multiplicity*** of a set of monomials is defined by

$$\mathrm{mult}(\mathcal{M}) = \max_{|S| \geq 2} \mathrm{mult}_{\mathcal{M}}(S)$$

Preliminaries 2/2

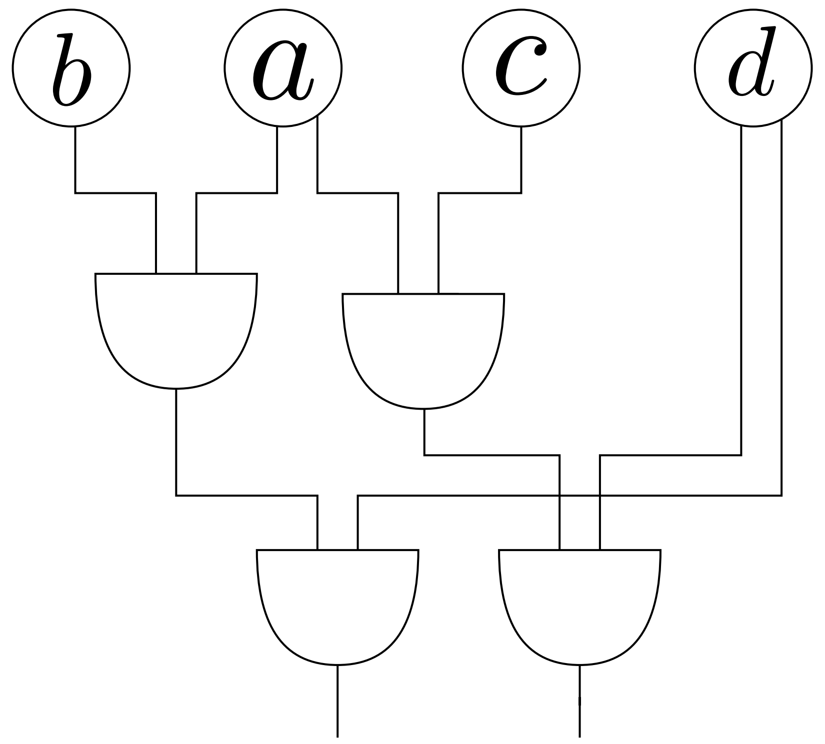
- An **AND-circuit** is a DAG that consists of **input nodes** and other nodes called **AND-*gates***
- Every **input node** can have in-degree 0 and arbitrary out-degree. Every **AND-*gate*** has in-degree 2 and arbitrary out-degree

Input node 

AND-gate 

- A circuit C computes a set of monomials $\mathcal{M}$ if all monomials in $\mathcal{M}$ are properly evaluated in C

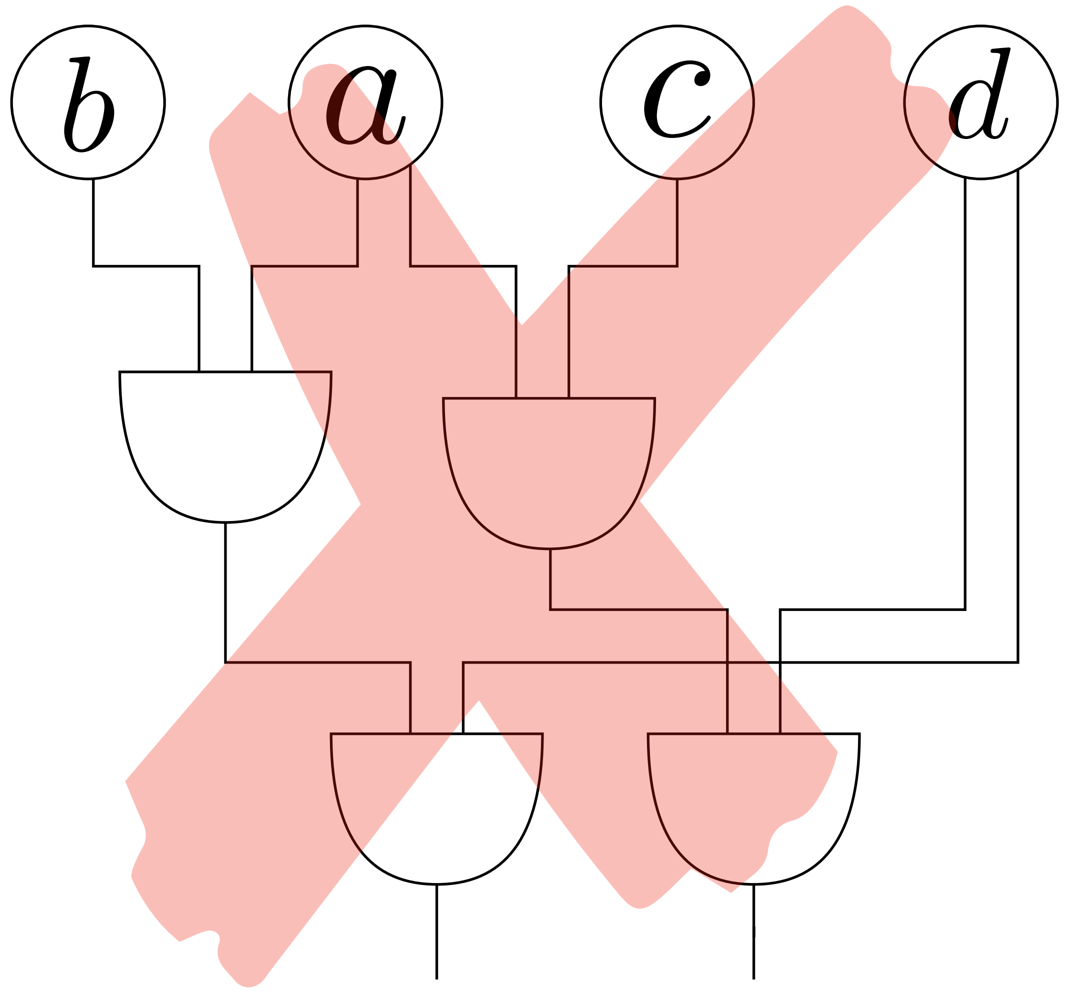
$$\mathcal{M} = \{M_1 : (a \wedge b \wedge d), M_2 : (a \wedge c \wedge d)\}$$

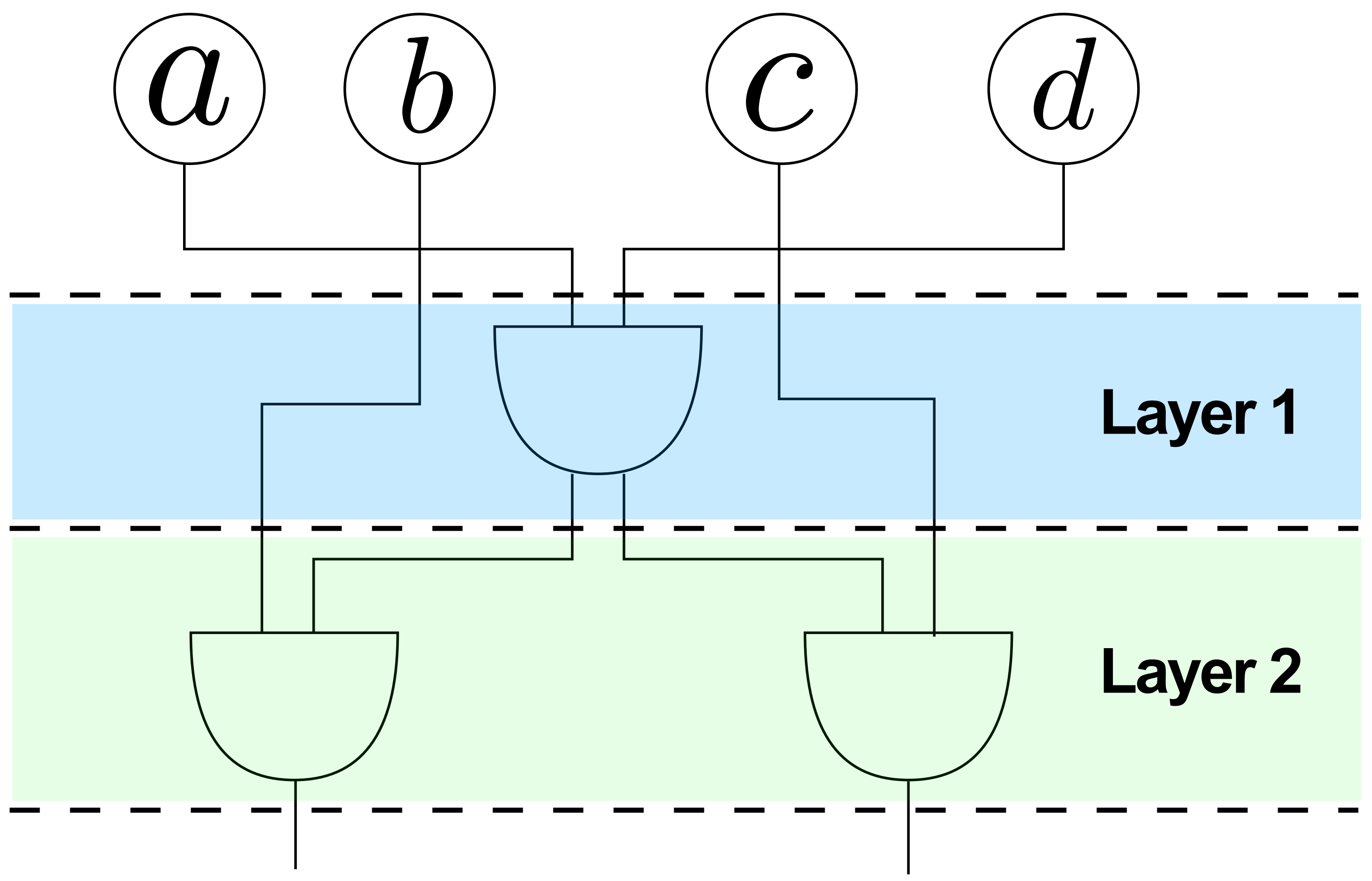


Minimum AND-Circuit

Given a set of monomials $\mathcal{M} = \{M_1, \ldots, M_k\}$ over a set of Boolean input variables $X = \{x_1, \ldots, x_n\}$ we aim to **find a circuit of minimum size which computes** $\mathcal{M}$

$$\mathcal{M} = \{M_1 : (a \wedge b \wedge d), M_2 : (a \wedge c \wedge d)\}$$





Algorithm Cover-μ

Algorithm $\textsc{Cover}_\mu$ for Min-3-AC with maximum multiplicity μ.

1: Input $\mathcal{M} = \{M_1, \dots, M_k\}$.
2: Compute the μ-Set Cover instance $\mathcal{S}(\mathcal{M})$.
3: Compute a set cover for $\mathcal{S}(\mathcal{M})$.
4: Output the circuit corresponding to the set cover.

w.l.o.g. monomials have degree 3

How the reduction to μ-Set Cover works:

- The universal set U is $\mathcal{M}$
- For each pair S in $\bigcup_{M \in \mathcal{M}} \binom{M}{2}$ add the set $\{M \in \mathcal{M} \mid S \in \binom{M}{2}\}$ to a collection C
- Since the multiplicity is at most μ, (U, C) is a μ-Set Cover instance

Algorithm Cover-μ

$$\mathcal{M} = \{M_1 : (a \wedge b \wedge d), M_2 : (b \wedge d \wedge e), M_3 : (b \wedge c \wedge d), M_4 : (a \wedge d \wedge e)\}$$

$$U = \mathcal{M}$$

$$\binom{M_1}{2} = \{(a,b),(b,d),(a,d)\}$$

$$\binom{M_2}{2} = \{(b,d),(d,e),(b,e)\}$$

$$\binom{M_3}{2} = \{(b,c),(c,d),(b,d)\}$$

$$\binom{M_4}{2} = \{(a,d),(d,e),(a,e)\}$$

3-Set Cover

$$C = \{(a,b) : \{M_1\}, (b,d) : \{M_1, M_2, M_3\}, (a,d) : \{M_1, M_4\},$$

$$(d,e) : \{M_2, M_4\}, (b,e) : \{M_2\}, (b,c) : \{M_3\}, (c,d) : \{M_3\}, (a,e) : \{M_4\}\}$$

Approximation

- Let k be the number of gates on the **second** layer of the optimal circuit
- Let ℓ be the number of gates on the **first** layer of the optimal circuit

Lemma 3.1. *For $\mu \geq 3$, $\textsc{Cover}_\mu$ outputs a circuit for $\mathcal{M}$ of size at most $k + (H_\mu - \frac{1}{2})\ell$, where $H_\mu = \sum_{i=1}^{\mu} \frac{1}{i}$.*

$$\rho_{\textsc{Cover}_\mu} \leq \frac{k + (H_\mu - \frac{1}{2})\ell}{k + \ell}, \quad \text{increasing in } \ell.$$

Theorem 3.2. *The* Min-3-AC *problem restricted to instances of maximum multiplicity three and four is approximable with a factor of* 1.125 *and* $45/38 < 1.185$*, respectively.*

Algorithm Greedy-μ

Algorithm $\textsc{Greedy}_\mu$ for Min-3-AC with unbounded multiplicity.

1: Input $\mathcal{M} = \{M_1, \ldots, M_k\}$.
2: While there exists an $S \in \binom{X}{2}$ such that $|\{M \in \mathcal{M} \mid S \subseteq M\}| \geq \mu + 1$:
3: Arbitrarily select $S \in \binom{X}{2}$ with maximum $|\{M \in \mathcal{M} \mid S \subseteq M\}|$.
4: Add a gate computing S to $\mathcal{C}$. → **k1/(μ+1) gates**
5: For each $M \in \mathcal{M}$ with $S \subseteq M$:
6: Add a gate computing M to $\mathcal{C}$. → **k1 gates**
7: $\mathcal{M} \leftarrow \mathcal{M} \setminus \{M\}$.
8: $\mathcal{C}' \leftarrow \textsc{Alg}_\mu(\mathcal{M})$. → **ρ(k2 + ℓ) gates**
9: $\mathcal{C} \leftarrow \mathcal{C} \cup \mathcal{C}'$
10: Output $\mathcal{C}$.

k1 monomials (lines 2–7)

k2 monomials (lines 8–9)

k1/(μ+1)+k1+ρ(k2 + ℓ) = $\frac{\mu+2}{\mu+1}k_1 + \rho_\mu(k_2 + \ell)$

Approximation

Lemma 3.3. *For $\mu \geq 2$, $\textsc{Greedy}_\mu$ outputs a circuit for $\mathcal{M}$ of size at most $\max\{\frac{\mu+2}{\mu+1}, \rho_\mu\}k + \rho_\mu \ell$.*

$$\frac{\mu+2}{\mu+1}k_1 + \rho_\mu(k_2 + \ell) \leq \max\left\{\frac{\mu+2}{\mu+1}, \rho_\mu\right\} k + \rho_\mu \ell$$

$$\rho_{\textsc{Greedy}_\mu} \leq \frac{\max\{\frac{\mu+2}{\mu+1}, \rho_\mu\}k + \rho_\mu \ell}{k+\ell}$$

$$\rho_{\textsc{Greedy}_4} \leq \frac{\frac{6}{5}k + \frac{45}{38}\ell}{k+\ell}, \quad \text{decreasing in } \ell.$$

Theorem 3.4. *The* Min-3-AC *problem is approximable with a factor of*

$$\frac{231e^2 - 225}{193e^2 - 190} < 1.199.$$

$$\rho_{\textsc{Greedy}} \leq \frac{(1+e^{-2})k + 2\ell}{k+\ell}, \quad \text{increasing in } \ell.$$

Conclusion

Table 1: Approximation ratios for Min-3-AC

maximum multiplicity	2	3	4	5	$\cdots$	unbounded
Arpe and Manthey [1]	P	1.25	$1.2\overline{6}$	$1.2\overline{7}$		1.278
Our results	-	1.125	1.185	1.198		1.199

References

H. Morizumi, “Improved approximation algorithms for minimum AND-circuits problem via k-set cover, Information Processing Letters”, Information Processing Letters, 111(5), pp. 218-221, 2011.

J. Arpe and B. Manthey, “Approximability of minimum AND-circuits”, Algorithmica 53(3), pp. 337–357, 2009.

Thank you!