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# LABELLING GRAPHS WITH A CONDITION AT DISTANCE 2*

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**Abstract.** Given a simple graph $G = (V, E)$ and a positive number $d$, an $L_d(2,1)$-*labelling* of $G$ is a function $f : V(G) \to [0, \infty)$ such that whenever $x, y \in V$ are adjacent, $|f(x) - f(y)| \geq 2d$, and whenever the distance between $x$ and $y$ is two, $|f(x) - f(y)| \geq d$. The $L_d(2,1)$-*labelling number* $\lambda(G,d)$ is the smallest number $m$ such that $G$ has an $L_d(2,1)$-labelling $f$ with $\max\{f(v) : v \in V\} = m$.

It is shown that to determine $\lambda(G,d)$, it suffices to study the case when $d = 1$ and the labelling is nonnegative integral-valued. Let $\lambda(G) = \lambda(G,1)$. The labelling numbers of special classes of graphs, e.g., $\lambda(C) = 4$ for any cycle $C$, are described. It is shown that for graphs of maximum degree $\Delta$, $\lambda(G) \leq \Delta^2 + 2\Delta$. If $G$ is diameter 2, $\lambda(G) \leq \Delta^2$, a sharp bound for some $\Delta$. Determining $\lambda(G)$ is shown to be NP-complete by relating it to the problem of finding Hamilton paths.

**Key words.** $T$-coloring, channel assignments, graph coloring, NP-completeness

**AMS(MOS) subject classifications.** 05C15, 05C35, 05C78, 05C85, 68R10

## 1. Introduction.

There has been a considerable effort (cf. [CR], [CW], [FGK], [G], [H], [R1], [R2], [Ro1], and [T]) to study properties of "T-colorings" of graphs, which is motivated by the task of assigning channel frequencies without interference. Roberts [Ro2] proposed the problem of efficiently assigning radio channels to transmitters at several locations, using nonnegative integers to represent channels, so that close locations receive different channels, and channels for very close locations are at least two apart. Therefore these channels would not interfere with each other.

We propose an analogous problem for simple graph $G = (V, E)$. Given a real number $d > 0$, an $L_d(2,1)$-*labelling* of $G$ is a nonnegative real-valued function $f : V(G) \to [0, \infty)$ such that, whenever $x$ and $y$ are two adjacent vertices in $V$, then $|f(x) - f(y)| \geq 2d$, and, whenever the distance between $x$ and $y$ is 2, then $|f(x) - f(y)| \geq d$. The $L_d(2,1)$-*labelling number* of $G$ is the smallest number $m$ such that $G$ has an $L_d(2,1)$-labelling with no label greater than $m$ and is denoted by $\lambda(G,d)$. If $f$ is an $L_d(2,1)$-labelling of $G$, then we say that $f \in L_d(2,1)(G)$.

Let $G$ be a graph and $f \in L_d(2,1)(G)$. Define $\|f(G)\| = \max\{f(v) : v \in V(G)\}$. Then $\lambda(G,d) = \min \|f(G)\|$, where the minimum runs over all $f \in L_d(2,1)(G)$. In the language of Roberts [Ro3], we are trying to minimize the *span* of an $L_d(2,1)$-labelling. However, we allow 0 to be a label, unlike most other analogous parameters, because we can then nicely characterize $\lambda(G,d)$ in terms of $\lambda(G,1)$. We describe this in §2, where we also show that for $\lambda(G,1)$ it suffices to consider integral-valued labellings. Thereafter we confine our study to $\lambda(G,1)$, which we denote simply by $\lambda = \lambda(G)$. Similarly, $L(2,1) = L(2,1)(G)$ denotes $L_1(2,1)(G)$. We let $[0,k]$ denote the set $\{0, 1, \ldots, k\}$.

In §§3–5 we consider the labelling numbers of some fundamental classes of graphs. In §6 we present general upper bounds on $\lambda$ in terms of the maximum degree $\Delta$. We

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find that $\lambda$ is never much larger than $\Delta^2$. Diameter 2 graphs are studied in the next section, and the sharp upper bound $\Delta^2$ is obtained for $\lambda$ in this case. Infinite families of graphs with $\lambda$ close to $\Delta^2$ are described in §8. After investigating the complexity of the $L_1(2,1)$-labelling problem in §9, we conclude by proposing some problems for further research.

**2. Reduction to integral-valued labellings.** First, we want to characterize $\lambda(G,d)$ in terms of $\lambda(G,1)$. Furthermore, we show that to determine $\lambda(G,1)$ it suffices to study the case when the labelling is integral-valued.

LEMMA 2.1. *It holds that $\lambda(G,d) = d \cdot \lambda(G,1)$.*

*Proof.* We prove the lemma with the following claims.

*Claim* 1. We have that $\lambda(G,d) \geq d \cdot \lambda(G,1)$.

Let $f \in L_d(2,1)(G)$. Define $f_1(x) = f(x)/d$, for all $x \in V(G)$. It follows easily that $f_1 \in L_1(2,1)(G)$. This implies that $\|f(G)\|/d = \|f_1(G)\| \geq \lambda(G,1)$. By compactness, some $f$ attains $\lambda(G,d)$, and the claim follows.

*Claim* 2. We have that $\lambda(G,d) \leq d \cdot \lambda(G,1)$.

The proof is similar to Claim 1. Therefore the result follows. ◻

LEMMA 2.2. *Let $x, y \geq 0$, $d > 0$ and $k \in Z^+$. If $|x-y| \geq kd$, then $|x'-y'| \geq kd$, where $x' = \lfloor x/d \rfloor d$ and $y' = \lfloor y/d \rfloor d$.*

The two lemmas above imply the following theorem.

THEOREM 2.3. *Given a graph $G$, there is an $f \in L_1(2,1)(G)$ such that $f$ is integral-valued and $\|f(G)\| = \lambda(G,1)$.*

For general $d$, we see that $\lambda(G,d)$ is attained by some $f \in L_d(2,1)(G)$ whose values are all multiples of $d$, i.e., $f = d \cdot f'$, where $f' \in L_1(2,1)(G)$ is integral-valued (by Lemma 2.1). Therefore it suffices to study the case where $d = 1$ and to consider in what follows only integral-valued $f \in L_1(2,1)(G)$.

**3. Paths, cycles, and cubes.** First, let us look at the $L(2,1)$-labelling of an elementary graph, the path. We have the following easy result (cf. [Y]).

PROPOSITION 3.1. *Let $P_n$ be a path on $n$ vertices. Then* (i) $\lambda(P_2) = 2$, (ii) $\lambda(P_3) = \lambda(P_4) = 3$, *and* (iii) $\lambda(P_n) = 4$, *for* $n \geq 5$.

If we join the first vertex and the last vertex of a path, then we have a cycle. So what is the labelling number of a cycle?

PROPOSITION 3.2. *Let $C_n$ be a cycle of length $n$. Then $\lambda(C_n) = 4$, for any $n$.*

*Proof.* If $n \leq 4$, then it is easy to verify the result. Thus suppose that $n \geq 5$. For all $n \geq 5$, $C_n$ must contain a $P_5$ as a subgraph. Hence $\lambda(C_n) \geq \lambda(P_5) = 4$, by Proposition 3.1.

Now we want to show that $\lambda(C_n) \leq 4$, $n \geq 5$. It suffices to show that there is an $L(2,1)$-labelling $f$ such that $\|f(C_n)\| = 4$. Let $v_0, \ldots, v_{n-1}$ be vertices of $C_n$ such that $v_i$ is adjacent to $v_{i+1}$, $0 \leq i \leq n-2$ and $v_0$ is adjacent to $v_{n-1}$. Consider the following labelling:

(1) If $n \equiv 0 \pmod 3$, then define

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 3, \\ 2, & \text{if } i \equiv 1 \pmod 3, \\ 4, & \text{if } i \equiv 2 \pmod 3; \end{cases}$$

(2) If $n \equiv 1 \pmod 3$, then redefine the above $f$ at $v_{n-4}, \ldots, v_{n-1}$ as

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$$f(v_i) = \begin{cases} 0, & \text{if } i = n-4, \\ 3, & \text{if } i = n-3, \\ 1, & \text{if } i = n-2, \\ 4, & \text{if } i = n-1; \end{cases}$$

(3) If $n \equiv 2 \pmod 3$, then redefine the $f$ in (1) at $v_{n-2}$ and $v_{n-1}$ as

$$f(v_i) = \begin{cases} 1, & \text{if } i = n-2, \\ 3, & \text{if } i = n-1. \end{cases}$$

It is easy to show that $f$, defined above, is in $L(2,1)(C_n)$ for every $n$, for each case. Hence $\lambda(C_n) \le 4$. Therefore the theorem is proved. ■

If we take a cycle $C_n$ joined by a vertex, then we have a graph $W_n$ called a *wheel of length* $n$, i.e., $W_n = C_n \vee K_1$. In [Y] it is shown that $\lambda(W_n) = n+1$.

Next, consider the *n-cube* $Q_n$, which has $2^n$ vertices $v = (v_1, \ldots, v_n)$, where each $v_i$ is 0 or 1, and edges join vertices $v, w$ when there exists a unique $i$ such that $v_i \ne w_i$. This bipartite graph is regular of degree $n$.

THEOREM 3.3. *Let $Q_n$ be the n-cube. Then, for all $n \ge 5$, $n+3 \le \lambda(Q_n) \le 2n+1$.*

*Proof.* The following modular labelling implies the stated upper bound for $n \ge 1$:

$$f(v) = \sum_{i: v_i = 1} (i+1) \qquad (\text{mod } 2n+2),$$

where all labels are chosen to belong to $[0, 2n+1]$. To verify this, consider adjacent vertices $v$ and $w$. We may assume that $v_i - w_i = 1$ (0, respectively) when $i = a$ ($i \ne a$, respectively). Then $f(v) - f(w) \equiv a + 1 \pmod{2n+2}$, so that $|f(v) - f(w)| \ge 2$. Similarly, if $v$ and $w$ are at distance 2, we may assume that $v_i - w_i$ is 1 when $i = a$, 1 or -1 when $i = b$, and 0 otherwise. Then $f(v) - f(w) \equiv a+b+2$ or $a - b \pmod{2n+2}$, so that $f(v) \ne f(w)$.

The lower bound of $n+3$ is due to Jonas [J]. Suppose for contradiction that $\lambda(Q_n) \le n+2$ for some $n \ge 5$ and let $f$ be an optimal labelling for such $Q_n$. Some vertex $v$ is labelled 0 in an optimal labelling. The $n$ vertices adjacent to $v$ receive labels that are distinct and greater than 1; i.e., each of $2, 3, \ldots, n+2$ is used with just one exception $i$. Since $n \ge 5$, if the labelling $f$ does not use the label 3, it must use the label $n-1$. In the later case, we may "reflect" $f$ and instead consider another optimal labelling, $n+2-f$. By permuting vertices, we may assume that our optimal labelling $f$ assigns 3 to vertex $w$. Let $W_i$ denote the set of vertices at distance $i$ from $w$. The vertices in $W_1$ must receive the distinct labels $0, 1, 5, 6, \ldots, n+2$. There are $\binom{n}{2}$ vertices in $W_2$, each adjacent to two vertices in $W_1$. If $x \in W_2$ is adjacent to the vertex in $W_1$ with label $j$, then $f(x) \ne j-1, j, j+1$. Two vertices in $W_2$ with the same label have no neighbors in common. It follows that label $i$ is used on $W_2$ at most $\lfloor (n-2)/2 \rfloor$ times when $i = 0, 1, 5, n+2$; $\lfloor (n-1)/2 \rfloor$ times when $i = 2, 4$; $\lfloor (n-3)/2 \rfloor$ times when $i = 6, \ldots, n+1$. Label 3 cannot be used on $W_2$. Adding up the possible labels does not account for all $\binom{n}{2}$ vertices in $W_2$, a contradiction. ■

With considerable effort, we have determined the first several values as follows: $\lambda(Q_0) = 0$, $\lambda(Q_1) = 2$, $\lambda(Q_2) = 4$, $\lambda(Q_3) = 6$, $\lambda(Q_4) = 7$, $\lambda(Q_5) = 8$. No pattern is yet evident in the labellings that attain these values. Jonas recently showed that $\lambda(Q_n) \ge n+4$ for $n = 8$ and 16. Using methods from coding theory, it was recently shown by Jonas and by Georges, Mauro, and Whittlesey that $\liminf_{n\to\infty} \lambda(Q_n)/n = 1$.

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**4. Trees.** We next discuss the labelling numbers of connected graphs without cycles, that is, trees. The maximum degree nearly determines the labelling number.

THEOREM 4.1. *Let $T$ be a tree with maximum degree $\Delta \geq 1$. Then $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$.*

*Proof.* Since $T$ contains the star $K_{1,\Delta}$, we have $\lambda(T) \geq \lambda(K_{1,\Delta}) = \Delta + 1$. We obtain the upper bound by a first-fit (greedy) labelling. First, order $V(T)$ so that $V(T) = \{v_1, \ldots, v_n\}$, where, for all $i > 1$, $v_i$ is attached just once to $\{v_1, \ldots, v_{i-1}\}$. This can be done since $T$ is a tree. Now we describe an $L(2,1)$-labelling of $T$: Label $v_1$ as 0; then successively label $v_2, v_3, \ldots, v_n$ by the lowest available element of $[0, \Delta + 2]$. Since each $v_i$, $2 \leq i \leq n$, is adjacent to only one $v_j$, $j < i$ and is distance 2 away from at most $\Delta - 1$ $v_j$'s with $j < i$, there are at most $\Delta + 2$ labels that cannot be used for $v_i$. Hence at least one label in $[0, \Delta + 2]$ is available to $v_i$ when its turn comes to be labeled. Thus the labelling number is at most $\Delta + 2$, and the theorem follows. ∎

$\Delta + 2 = 5$

$\Delta = 3$

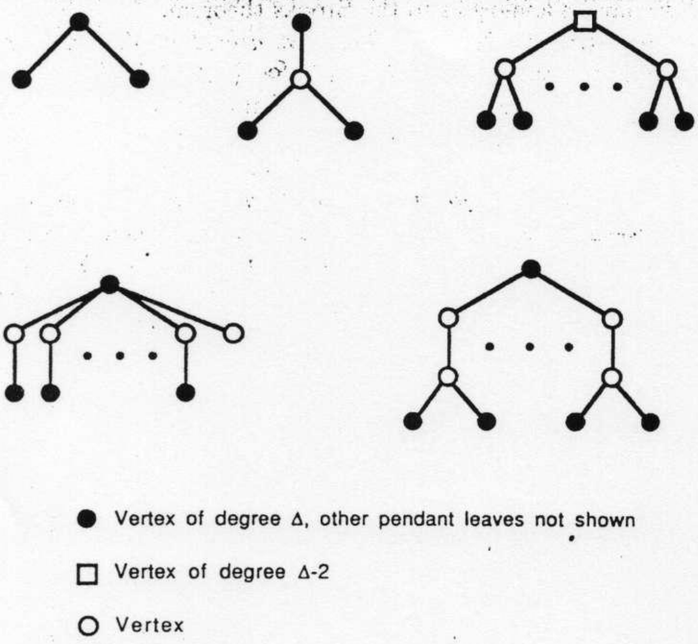


FIG. 1. *Critical trees with labelling number $\Delta + 2, \Delta \geq 3$.*

Both values can occur. The value $\Delta + 1$ holds for many trees, e.g., the star $K_{1,\Delta}$. We exhibit several trees in Fig. 1 with $\lambda = \Delta + 2$. All trees shown are, in fact, $\lambda$-critical; i.e., deleting any vertex (or edge) drops $\lambda$. It seems that characterizing all trees with $\lambda = \Delta + 2$ is very difficult (see §10).

**5. $k$-colorable graphs.** Before considering the labelling number of general graphs $G$, we want to look at graphs with specified chromatic number $\chi(G)$.

THEOREM 5.1. *Let $G$ be a graph with $\chi(G) = k$ and $|V(G)| = \nu$. Then $\lambda(G) \leq \nu + k - 2$.*

*Proof.* Since $\chi(G) = k$, we can partition $G$ into $G_1 \cup \cdots \cup G_k$, where $|V(G_i)| = \nu_i$ and each $G_i$ is an independent set. Let $V_i = V(G_i) = \{v_{i,1}, v_{i,2}, \ldots, v_{i,\nu_i}\}$, $1 \leq i \leq k$. Now consider the labelling $f$ defined by

$$f(v_{1,j}) = j - 1, \qquad 1 \leq j \leq \nu_1,$$

$$f(v_{i,j}) = \sum_{l=1}^{i-1} \nu_l + i + j - 2 \qquad \begin{cases} 1 \leq j \leq \nu_i \\ 2 \leq i \leq k \end{cases}$$

$\Rightarrow$ ... $= \nu + k - 2$ ∎

$$f(v_{i,j}) = \sum_{t=1}^{i-1} \nu_t + i + j - 2, \qquad 1 \le j \le \nu_i, \qquad \text{for } 2 \le i \le k.$$

It is easily verified that $f$ is in $L(2,1)(G)$. Hence $\lambda(G) \le \|f(G)\| = \nu + k - 2$. □

COROLLARY 5.2. *Let $G$ be a complete $k$-partite graph with $|V(G)| = \nu$. Then $\lambda(G) = \nu + k - 2$.*

*Proof.* Since $G$ is a $k$-partite graph, $\chi(G) = k$. By Theorem 5.1, $\lambda(G) \le \nu + k - 2$. On the other hand, since the distance between any two vertices in $G$ is at most 2, the labels must be distinct. Furthermore, consecutive labels cannot be used at vertices from different parts. Since we have $k$ components, we find that $\lambda(G) \ge \nu + k - 2$, and the result follows. □

**6. Upper bounds on $\lambda$ in terms of the maximum degree.** In this section, we determine the upper bound on $\lambda(G)$ in terms of the maximum degree of $G$. The upper bound we have is analogous to the Brooks theorem.

THEOREM 6.1 (Brooks [Br]). *Let $G$ be a connected graph of maximum degree $\Delta$. If $G$ is not a complete graph or an odd cycle, then $\chi(G) \le \Delta$.* □

Before showing the result, whose proof is analogous to that of Theorem 6.1, we give the following simple result, which uses a first-fit (greedy) labelling to provide a bound on $\lambda$ in terms of the maximum degree $\Delta$ for any graph.

THEOREM 6.2. *Let $G$ be a graph with maximum degree $\Delta$. Then $\lambda(G) \le \Delta^2 + 2\Delta$.*

*Proof.* Arbitrarily order the vertices of $G$, and label them in succession by the lowest allowed integer. A vertex $v \in V$ is adjacent to at most $\Delta$ vertices, and there are at most $\Delta^2 - \Delta$ vertices, which are distance 2 away from $v$. So, when we want to label $v$, there are at most $3\Delta + \Delta^2 - \Delta = \Delta^2 + 2\Delta$ numbers to be avoided. Thus the labelling number $\lambda(G)$ is at most $\Delta^2 + 2\Delta$. (Since we can use 0 to label a vertex, there are $\Delta^2 + 2\Delta + 1$ numbers that can be used.) □

We can improve the bound above when $G$ is 3-connected. The argument of the next theorem is analogous to the proof, which is due to Lovász [BM] of the Brooks theorem.

THEOREM 6.3. *If $G$ is a 3-connected graph, then $\lambda(G) \le \Delta^2 + 2\Delta - 3$.*

*Proof.* If $G$ is complete, then it is trivial, since it is easy to see that $\lambda(G) = 2\Delta$. Suppose that $G$ is not complete. Then there exist three vertices $u, v, w$ in $V$ such that $\{u, v\}$ and $\{v, w\}$ are in $E$ but $\{u, w\}$ is not in $E$. Set $v_1 = u$ and $v_2 = w$ and let $v_3, v_4, \ldots, v_\nu = v$ ($\nu = |V|$) be any ordering of the vertices of $V - \{u, w\}$ such that each $v_i, 3 \le i \le \nu - 1$ is adjacent to some $v_j$ with $j > i$; e.g., order the vertices by nonincreasing distance from $v$ in $G - \{u, w\}$. We can now describe an $L(2,1)$-labelling of $G$: Label $v_1$ as 0 and $v_2$ as 1; then successively label $v_3, v_4, \ldots, v_\nu$ with the lowest available label $\ge 0$. Each vertex $v_i$, $1 \le i \le \nu - 1$ is adjacent to at most $\Delta - 1$ vertices $v_j$ with $j < i$. Each such $v_j$ eliminates at most three possible choices for the label at $v_i$. Furthermore, there are at most $\Delta(\Delta - 1)$ vertices $v_k$, $k < i$, at distance 2 from $v_i$, and each such $v_k$ eliminates at most one choice for the label at $v_i$. It follows that, when its turn comes to be labeled, some label in $[0, \Delta^2 + 2\Delta - 3]$ will be available for $v_i$. Finally, since $v_\nu = v$ is adjacent to two vertices with labels 0 and 1, there is some label in $[0, \Delta^2 + 2\Delta - 3]$ available for $v_\nu$. □

We can show that the first-fit labelling given in the above proof uses label $\Delta^2 + 2\Delta - 3$ at most once, so it is likely that a more careful argument can improve the bound.

Sakai [S] observed that the bound in Theorem 6.2 can be improved for chordal graphs, which are graphs that contain no induced cycles of length at least 4. The idea is order the vertices carefully: A chordal graph has an ordering $\{v_1, v_2, \dots\}$ of $V$ such that, for all $i$, the neighbors of $v_i$ among $\{v_1, \dots, v_{i-1}\}$ form a clique. An analysis of the first-fit coloring for this sequence yields her result.

THEOREM 6.4 (Sakai [S]). *Let $G$ be a chordal graph with maximum degree $\Delta$. Then $\lambda(G) \le (\Delta+3)^2/4$.* ⇒ [3]

**7. Diameter 2 graphs.** We have a better upper bound for a class of graphs that is important in our study, namely, the diameter 2 graphs. The upper bound for this case, $\Delta^2$, is sharp for some $\Delta$.

Now we present the following lemma, which will allow us to prove the result mentioned above and to determine the complexity of the $L(2,1)$-labelling problem in the next section.

LEMMA 7.1. *The following two statements are equivalent:*

(1) *There exists an injection $f : V(G) \to [0, |V|-1]$ such that $|f(x)-f(y)| \ge 2$ for all $\{x,y\} \in E(G)$;*

(2) *$G^c$ contains a Hamilton path.*

*Proof.* (1) ⇒ (2): Let $f$ be an injection defined on $V$ that satisfies the condition in (1). Since $f$ is injective, $f^{-1}$ exists. Order the vertices of $V$ as follows: $v_i = f^{-1}(i)$, $0 \le i \le |V|-1$. Then $v_i$ is adjacent to $v_{i+1}$ in $G^c$ for $0 \le i \le |V|-1$. Therefore the path $\{v_0, v_1, \dots, v_{|V|-1}\}$ is a Hamilton path of $G^c$.

(2) ⇒ (1): Let $P = \{v_0, v_1, \dots, v_{|V|-1}\}$ be a Hamilton path of $G^c$. Define the function $f : V(G) \to [0, |V|-1]$ by $f(v_i) = i$, $0 \le i \le |V|-1$. Then it is easy to see that $f$ is injective. Let $\{x,y\} \in E(G)$. Then $f(x) = f(v_i) = i$ and $f(y) = f(v_j) = j$ for some $i, j$ with $|i-j| \ge 2$ since $x$ is not adjacent to $y$ in $G^c$. Hence $f$ is the injection we need. □

To prove Theorem 7.3, we also need the following theorem due to Dirac [D] (see also [BM]).

THEOREM 7.2. *Let $G$ be a simple graph with $|V| \ge 3$ and minimum degree $\delta \ge |V|/2$. Then $G$ is Hamiltonian.*

Now we present our bound for diameter 2 graphs.

THEOREM 7.3. *If $G$ is a graph with diameter 2, then $\lambda(G) \le \Delta^2$.*

*Proof.* If $\Delta = 2$, then we can verify the result directly, since, in this case, $G$ is either $C_4$, $C_5$ or a path of length 2. Thus assume that $\Delta \ge 3$.

Suppose that $\Delta \ge (|V|-1)/2$. By Theorem 5.1, we have that $\lambda \le |V| + \chi - 2 \le 2\Delta + 1 + \Delta - 2 = 3\Delta - 1 < \Delta^2$, since $\Delta \ge 3$.

Now suppose that $\Delta < (|V|-1)/2$. Then $\delta(G^c) \ge |V|/2$. Since diam$(G)$=2, obviously $|V| \ge 3$. Hence, by Theorem 7.2, $G^c$ is Hamiltonian; i.e., $G^c$ contains a Hamilton path. By Lemma 7.1, there is an injection $f : V \to [0, |V|-1]$ such that $|f(x) - f(y)| \ge 2$, for all $\{x,y\} \in E(G)$. From here, it is easy to see that $f \in L(2,1)(G)$ and $\|f(G)\| = |V|-1$. $G$ is a diameter 2 graph, so $|V| \le \Delta^2 + 1$. Therefore $\lambda(G) \le |V|-1 \le \Delta^2$. □

Note that the upper bound $\Delta^2$ is the best possible only when $\Delta = 2, 3, 7$, and possibly 57 because a diameter 2 graph with $|V| = \Delta^2 + 1$ can exist only if $\Delta$ is one of these numbers (cf. [HS]). When $\Delta = 2$, the graph is $C_5$; when $\Delta = 3$, it is the *Petersen graph*. For the graph when $\Delta = 7$, it is called the *Hoffman–Singleton graph* (see [HS] or [BM]). Since diam$(G)$=2, all labels in $V$ must be distinct. Hence $\lambda(G) \ge |V| - 1 = \Delta^2$. On the other hand, by Theorem 7.3, $\lambda \le \Delta^2$. Thus $\lambda(G) = \Delta^2$ only if $\Delta(G) = 2, 3, 7$, and possibly 57.

$\lambda_{2,1} \le 3\Delta$

(+1)



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According to the proof of Theorem 7.3, if $\Delta \geq 3$, we also know that $\lambda < \Delta^2$ whenever $|V| < \Delta^2+1$. Hence, in general, except for those extremal graphs mentioned above and $C_4$, whose labelling number also is $\Delta^2(=4)$, $\Delta^2 - 1$ is an upper bound on $\lambda$ for a diameter 2 graph.

**8. Two special classes of graphs.** In this section, we will present two classes of graphs with $\lambda$ that is close to the bound we have in Theorem 6.2.

First, let us give some definitions. We say a graph $G$ is an *incidence graph* of a projective plane $\Pi(n)$ of order $n$, if $G = (A, B, E)$ is a bipartite graph such that

(1) $|A| = |B| = n^2 + n + 1$,

(2) each $a \in A$ corresponds to a point $p_a$ in $\Pi(n)$ and each $b \in B$ corresponds to a line $\ell_b$ in $\Pi(n)$, and

(3) $E = \{\{a, b\} : a \in A,\ b \in B \text{ such that } p_a \in \ell_b \text{ in } \Pi(n)\}$.

By the definition of $\Pi(n)$, we know that such $G$ is $(n + 1)$-regular, for every $x, y \in A$, $d_G(x, y) = 2$, and for every $u, v \in B$, $d_G(u, v) = 2$. Also, if $a \in A$, $b \in B$ such that $a$ is not adjacent to $b$, then $d_G(a, b) = 3$. In [Y] we have the following theorem.

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THEOREM 8.1. *If $G$ is the incidence graph of a projective plane of order $n$, then $\lambda(G) = n^2 + n = \Delta^2 - \Delta$, where $\Delta = n + 1$, the maximum degree of $G$.*

Before the next theorem, let us recall the definition of the *Galois plane*. Let $K$ be the *Galois field of order* $n$ and let $P = \{(x_1, x_2, x_3) : x_i \in K\}\backslash\{(0, 0, 0)\}$. Define an equivalence relation $\equiv$ on $P$ in the following manner: $(x_1, x_2, x_3) \equiv (y_1, y_2, y_3)$ if and only if there exists $c \in K$, $c \neq 0$ for which $y_1 = cx_1$, $y_2 = cx_2$, $y_3 = cx_3$. Let these equivalence classes be called *points*. The set of all points defined by an equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$, where $a_1$, $a_2$, $a_3 \in K$ and not all are zero, will be called a *line*, which is denoted by $[a_1, a_2, a_3]$.

The projective plane determined above will be called a *Galois plane* (*over the coordinate field* $GF(n)$) and will be denoted by $PG_2(n)$ (cf. [K]).

Next, we construct another class of graphs from the Galois plane $PG_2(n)$ (cf. [B]). Let $V(H)$ be the set of points of $PG_2(n)$ and join a point $(x, y, z)$ to a point $(x', y', z')$ if $xx' + yy' + zz' = 0$, i.e., if $(x', y', z')$ lies on the line $[x, y, z]$. We call such a graph $H$ the *polarity graph* of $PG_2(n)$. Then by the properties of $PG_2(n)$, we know that $|V(H)| = n^2 + n + 1$, the maximum degree $\Delta(H) = n + 1$, the minimum degree $\delta(H) = n$ and the diameter is 2 (cf. [B]). Now we present the following theorem from [Y].

THEOREM 8.2. *If $H$ is the polarity graph of the Galois plane, $PG_2(n)$ then $\lambda(H) = n^2 + n = \Delta^2 - \Delta$, where $\Delta$ is the maximum degree of $H$.*

**9. The complexity of the $L(2,1)$-labelling problem.** It is well known that the coloring problem is an NP-complete problem. Since our $L(2, 1)$-labelling problem is similar to the coloring problem, we may guess it is also NP-complete. In this section, we verify this claim.

We need to consider the following special form of the $L(2, 1)$-labelling problem, where (DL) denotes distance 2 labelling:

(DL) *Instance*: Graph $G = (V, E)$ with diameter 2.
*Question*: Is $\lambda(G) \leq |V|$?

THEOREM 9.1. *(DL) is NP-complete.*

*Proof.* To show that (DL) is NP-complete, we study the following decision problem, where (IDL) denotes injective distance 2 labelling:

(IDL) *Instance:* Graph $G = (V, E)$.
*Question:* Is there an injection $f : V \to [0, |V| - 1]$ such that $|f(x) - f(y)| \geq 2$ whenever $\{x, y\} \in E$?

In view of Lemma 7.1, the NP-completeness of (IDL) follows as an immediate consequence of the well-known NP-completeness of the Hamilton path problem (HP) (see [GJ]) below:

(HP) *Instance:* Graph $G = (V, E)$.
*Question:* Is there a Hamilton path in $G$?

Next, we observe that (DL) is in NP. A graph $G = (V, E)$ can be input in time $O(|V| + |E|)$, and clearly we can verify in polynomial time that $G$ has diameter 2, that a labelling $f$ is in $L(2,1)(G)$, and that $\|f(G)\| \leq |V|$ .

We now show that (DL) is NP-complete by transformation from (IDL) to (DL). Let $G = (V, E)$ be any graph in the instance of (IDL). Construct a graph $G'$ as follows: Add a vertex $x$ to $V$ and let $x$ be adjacent to every vertex of $V$, i.e., $G' = (V', E')$, where $V' = V \cup \{x\}$ and $E' = E \cup \{\{x, v\} :$ for all $v \in V\}$. Then $|V'| = |V| + 1$ and $\mathrm{diam}(G')=2$.

The NP-completeness of (DL) then follows from the NP-completeness of (IDL) and from the following claim.

*Claim.* There is an injection $f : V(G) \to [0, |V| - 1]$ such that $|f(u) - f(v)| \geq 2$ for every $\{u, v\} \in E(G)$ if and only if $\lambda(G') \leq |V'|$.

*Proof of claim.* Suppose that there exists an injective function $f$ defined on $V$ that satisfies the condition above. Define $g(v) = f(v)$ for all $v \in V$ and $g(x) = |V| + 1 = |V'|$. Then easily $g \in L(2,1)(G')$ and $\|g(G')\| = |V| + 1 = |V'|$. Hence $\lambda(G') \leq |V'|$.

Conversely, suppose that $\lambda(G') \leq |V'|$, i.e., there is a $g$ in $L(2,1)(G')$ such that $\|g(G')\| \leq |V| + 1$. Suppose that $g(x) \neq 0$ or $|V| + 1$. By the property of $L(2,1)(G')$, there is no $v$ in $V$ such that $g(v) = g(x) + 1$ or $g(x) - 1$. This implies that we must use $|V| + 3$ numbers to label $V'$, which is a contradiction, since $\|g(G')\| \leq |V| + 1$ and all labels are distinct.

Hence $g(x)$ is either 0 or $|V| + 1$. If $g(x) = |V| + 1$, then restricting $g$ to $V$ gives the desired injection $f$. Similarly, if $g(x) = 0$, then restricting $g - 2$ to $V$ gives $f$. □

**10. Further research.** Inspired by the more general proximity-interference problems, a more general context would be to study labellings $f$, where $N$ is a positive integer and $m_1 \geq m_2 \geq \cdots \geq m_N > 0$ are given numbers. We require that $|f(x) - f(y)| \geq m_i$ if $d_G(x, y) = i$, $1 \leq i \leq N$. If $N = 1$ and $m_1 = 1$, then we have ordinary graph coloring. If $N = 2$, $m_1 = 2$, and $m_2 = 1$, then it is the $L(2,1)$-labelling. If $N = 2$, $m_1 = 1 = m_2$, then we have the $L(1,1)$-labelling, which has been studied in [Y].

Recall from the proof of Theorem 7.3 that, if $\Delta \geq 3$ and $\Delta \geq (|V| - 1)/2$, then we have $\lambda \leq \Delta^2$, regardless of whether $G$ has diameter 2. Therefore it is reasonable to propose the following conjecture.

CONJECTURE 10.1. *For any graph $G$ with maximum degree $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$.*

This conjecture holds for $\Delta = 2$ in view of Propositions 3.1 and 3.2.

In §9 we proved that the problem (DL) is NP-complete, but $\lambda$ was not a fixed value there. Consider the following decision problem for fixed $\lambda$, where (DLk) denotes distance 2 labelling with upper bound $k$:

(DLk) *Instance*: Graph $G = (V, E)$.
*Question*: Is $\lambda(G) \leq k$?

CONJECTURE 10.2. *For $k \geq 4$, (DLk) is NP-complete.*

For nontrivial trees $T$, we saw in §4 that $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$. It seems to be quite difficult to determine which of the two values holds, somewhat analogous to the situation for edge-colorings of graphs. Consider this decision problem for trees:

(TREE) *Instance*: Tree $T = (V, E)$.
*Question*: Is $\lambda(T) = \Delta(T) + 1$?

CONJECTURE 10.3. *The problem* (TREE) *is* NP-*complete.*

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REFERENCES

[B] B. BOLLOBÁS, *Extremal Graph Theory*, Academic Press, London, 1978.

[BM] J. A. BONDY AND U. S. R. MURTY, *Graph Theory with Applications*, North–Holland, New York, 1976.

[Br] R. L. BROOKS, *On coloring the nodes of a network*, Proc. Cambridge Philos. Soc., 37 (1941), pp. 194–197.

[CR] M. B. COZZENS AND F. S. ROBERTS, *T-colorings of graphs and the channel assignment problem*, Congr. Numer., 35 (1982), pp. 191–208.

[CW] M. B. COZZENS AND D. I. WANG, *The general channel assignment problem*, Congr. Numer., 41 (1984), pp. 115–129.

[D] G. A. DIRAC, *Some theorems on abstract graphs*, Proc. London Math. Soc., 2 (1952), pp. 69–81.

[FGK] Z. FÜREDI, J. R. GRIGGS, AND D. J. KLEITMAN, *Pair labellings with given distance*, SIAM J. Discrete Math., 2 (1989), pp. 491–499.

[G] M. C. GOLUMBIC, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.

[GJ] M. R. GAREY AND D. S. JOHNSON, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman, New York, 1979.

[H] W. K. HALE, *Frequency assignment: Theory and applications*, Proc. IEEE, 68 (1980), pp. 1497–1514.

[HS] A. J. HOFFMAN AND R. R. SINGLETON, *On Moore graphs with diameters 2 and 3*, IBM J. Res. Develop., 4 (1960), pp. 497–504.

[J] K. JONAS, private communication, 1991.

[K] F. KÁRTESZI, *Introduction to Finite Geometries*, North–Holland, New York, 1976.

[R1] A. RAYCHAUDHURI, *Intersection assignment, T-coloring and powers of graphs*, Ph.D. thesis, Department of Mathematics, Rutgers University, New Brunswick, NJ, 1985.

[R2] ———, *Further results on T-coloring and frequency assignment problems*, SIAM J. Discrete Math., submitted.

[Ro1] F. S. ROBERTS, *T-colorings of graphs: Recent results and open problems*, Research Report RRR 7-86, RUTCOR, Rutgers University, New Brunswick, NJ, 1986.

[Ro2] ———, private communication, 1988.