and so $\alpha(G_p) \geq k$. Let $S = \{v_{j_1}, v_{j_2}, \ldots, v_{j_k}\}$ be an independent set of k vertices of G_p such that $j_1 < j_2 < \cdots < j_k$.

Now let there be given a k-coloring of G, using the colors $1, 2, \ldots, k$. We now replace each color i $(1 \leq i \leq k)$ by j_i, arriving at a new k-coloring c of G. Hence if x and y are two adjacent vertices of G, then x and y are assigned distinct colors j_r and j_s, where $1 \leq r, s \leq k$ and $r \neq s$. Since $v_{j_r}, v_{j_s} \in S$, it follows that $v_{j_r}v_{j_s} \notin E(G_p)$ and so $|j_r - j_s| \notin T$. Hence c is a T-coloring of G. Since the largest color used in c is j_k and

$$j_k \leq p = \chi(G_s)(k-1) + 1,$$

it follows by (14.2) that

$$\begin{aligned} sp_T(G) &\leq j_k - j_1 \leq [\chi(G_s)(k-1) + 1] - 1 \\ &= \chi(G_s)(k-1) \leq t(k-1), \end{aligned}$$

giving the desired result. ■

We now show that the upper bound given in Theorem 14.4 for the T-span of a graph is attainable. Suppose first that $T = \{0, 2, 4\}$ and consider the graphs C_3 and C_5. Then $\chi(C_3) = \chi(C_5) = 3$ and $|T| = 3$. By Theorem 14.4, $sp_T(C_3) \leq 6$ and $sp_T(C_5) \leq 6$. Figure 14.4 shows T-colorings for these graphs with T-span 6. We show for $T = \{0, 2, 4\}$ that the T-span of C_3 is, in fact, 6. Assume, to the contrary, that $sp_T(C_3) = a$, where $a \leq 5$. Then there exists a T-coloring of C_3, where some vertex u of C_3 is colored 1 and the largest color assigned to a vertex v of C_3 is $a + 1 \leq 6$. Since $T = \{0, 2, 4\}$, either $a = 2$ or $a = 4$, and the color of the remaining vertex w of C_3 is of the same parity as either $c(u)$ or $c(v)$, which is impossible since $T = \{0, 2, 4\}$.

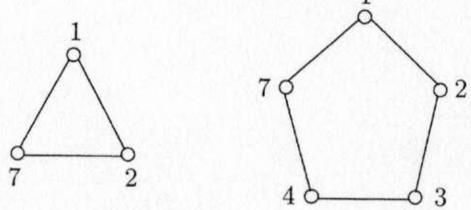

Figure 14.4: T-colorings of C_3 and C_5

An infinite class of graphs verifying the sharpness of the upper bound for $sp_T(G)$ stated in Theorem 14.4 consists of the complete graphs K_n with $T = \{0, 1, \ldots, t-1\}$, where $t \in \mathbb{N}$. By Theorem 14.4, $sp_T(K_n) \leq t(n-1)$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Assigning the color $t(i-1)+1$ to v_i for $i = 1, 2, \ldots, n$ gives a T-coloring of K_n. If $sp_T(K_n) < t(n-1)$, then there is a T-coloring of K_n where the difference in colors of two vertices is less than t. This, however, is impossible and so $sp_T(K_n) = t(n-1)$.

14.2 L(2, 1)-Colorings

One of the early types of colorings inspired by the Channel Assignment Problem occurred as a result of a communication to Jerrold Griggs by Fred Roberts, who proposed using nonnegative integers to represent radio channels in order to study the problem of optimally assigning radio channels to transmitters at certain locations. As a result of this, Roger Yeh [191] in 1990 and then Griggs and Yeh [83] in 1992 introduced a coloring in which colors (nonnegative integers in this case) assigned to the vertices of a graph depend not only on whether two vertices are adjacent but also on whether two vertices are at distance 2.

For nonnegative integers h and k, an $L(h,k)$**-coloring** c of a graph G is an assignment of colors (nonnegative integers) to the vertices of G such that if u and w are adjacent vertices of G, then $|c(u) - c(w)| \geq h$ while if $d(u,w) = 2$, then $|c(u) - c(w)| \geq k$. No condition is placed on colors assigned to u and v if $d(u,w) \geq 3$. Hence an $L(1,0)$-coloring of a graph G is a proper coloring of G. As with T-colorings, the major problems of interest with $L(h,k)$-colorings concern spans. For given nonnegative integers h and k and an $L(h,k)$-coloring c of a graph G, the **span** of c (or the c-**span** of G) is $\max|c(u) - c(w)|$ over all pairs u, w of vertices of G, which we denote by $\lambda_{h,k}(c)$. That is,

$$\lambda_{h,k}(c) = \max\{|c(u) - c(w)| : u, w \in V(G)\}.$$

For given nonnegative integers h and k, the $\lambda_{h,k}$**-number** or L**-span** of G is

$$\lambda_{h,k}(G) = \min\{\lambda_{h,k}(c)\}$$

where the minimum is taken over all $L(h,k)$-colorings c of G. Most of the interest in $L(h,k)$-colorings has been in the case where $h = 2$ and $k = 1$. Therefore, an $L(2,1)$-**coloring** of a graph G (also called an $L(2,1)$**-labeling** by some) is an assignment of colors (nonnegative integers, rather than the more typical positive integers) to the vertices of G such that

(1) colors assigned to adjacent vertices must differ by at least 2,

(2) colors assigned to vertices at distance 2 must differ, and

(3) no restriction is placed on colors assigned to vertices at distance 3 or more.

For an $L(2,1)$-coloring c of a graph G then, the c-**span** of G is

$$\lambda_{2,1}(c) = \max\{|c(u) - c(w)| : \ u, w \in V(G)\}.$$

For simplicity, the c-span $\lambda_{2,1}(c)$ of G is also denoted by $\lambda(c)$. The L-**span** or $\lambda_{2,1}$**-number** $\lambda_{2,1}(G)$ of G is therefore

$$\lambda_{2,1}(G) = \min\{\lambda(c)\},$$

where the minimum is taken over all $L(2,1)$-colorings c of G. Here too, many have simplified the notation $\lambda_{2,1}(G)$ to $\lambda(G)$. (Since $\lambda(G)$ is common notation for the edge-connectivity of a graph G, it is essential to know the context in which this symbol is being used.) Therefore, in this context, $\lambda(G)$ is the smallest positive integer k for which there exists an $L(2,1)$-coloring $c: V(G) \to \{0,1,\ldots,k\}$. Since we may always take 0 as the smallest color used in an $L(2,1)$-coloring of a graph G, it follows that $\lambda(G)$ is the smallest maximum color that can occur in an $L(2,1)$-coloring of G.

We determine $\lambda(G)$ for the graph G of Figure 14.5(a). The coloring c of G in Figure 14.5(b) is an $L(2,1)$-coloring and so $\lambda(c) = 5$. Hence $\lambda(G) \leq 5$. We claim that $\lambda(G) = 5$. Suppose that $\lambda(G) < 5$. Let c' be an $L(2,1)$-coloring such that $\lambda(c') = \lambda(G)$. We may assume that c' uses some or all of the colors $0, 1, 2, 3, 4$. Since the vertices u, v, and w are mutually adjacent, these three vertices must be colored $0, 2$, and 4, say $c'(u) = 0$, $c'(v) = 2$, and $c'(w) = 4$. Since $c'(y)$ must differ from $c'(v)$ by at least 2, it follows that $c'(y) = 0$ or $c'(y) = 4$. However, u and w are at distance 2 from y, implying that $c'(y) \neq 0$ and $c'(y) \neq 4$. This is a contradiction. Thus, as claimed, $\lambda(G) = 5$.

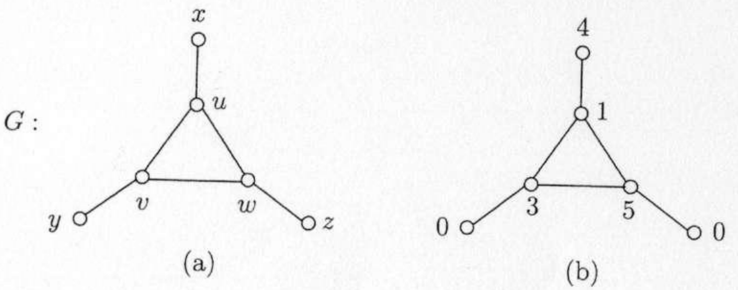

Figure 14.5: A graph G with $\lambda(G) = 5$

A family of graphs whose L-span is easy to determine are the stars.

Theorem 14.5 *For every positive integer t, $\lambda(K_{1,t}) = t + 1$.*

Proof. Since the result is immediate if $t = 1$, we may assume that $t \geq 2$. The coloring of $K_{1,t}$ that assigns $0, 1, \ldots, t-1$ to the t end-vertices of $K_{1,t}$ and $t+1$ to the central vertex of $K_{1,t}$ is an $L(2,1)$-coloring of $K_{1,t}$. Thus $\lambda(K_{1,t}) \leq t + 1$.

Suppose that there is an $L(2,1)$-coloring of $K_{1,t}$ using colors in the set $S = \{0, 1, \ldots, t\}$. Since the order of $K_{1,t}$ is $t+1$ and $\mathrm{diam}(K_{1,t}) = 2$, it follows that for each $i \in S$, exactly one vertex of $K_{1,t}$ is assigned the color i. In particular, the central vertex of $K_{1,t}$ is assigned a color $j \in S$. Because some end-vertex of $K_{1,t}$ must be colored $j-1$ or $j+1$, this coloring cannot be an $L(2,1)$-coloring of $K_{1,t}$. Hence we have a contradiction and so $\lambda(K_{1,t}) = t + 1$. ∎

The L-span of a tree with maximum degree Δ can only be one of two values.

Theorem 14.6 *If T is a tree with $\Delta(T) = \Delta \geq 1$, then either*

$$\lambda(T) = \Delta + 1 \text{ or } \lambda(T) = \Delta + 2.$$

Proof. Suppose that the order of T is n. Because $K_{1,\Delta}$ is a subgraph of T and $\lambda(K_{1,\Delta}) = \Delta + 1$ by Theorem 14.5, it follows that $\lambda(T) \geq \Delta + 1$. We now show that there exists an $L(2,1)$-coloring of T with colors from the set

$$S = \{0, 1, \ldots, \Delta + 2\}$$

of $\Delta + 3$ colors. Denote T by T_n and let v_n be an end-vertex of T_n. Let $T_{n-1} = T_n - v_n$ and let v_{n-1} be an end-vertex of T_{n-1}. We continue in this manner until we arrive at a trivial tree T_1 consisting of the single vertex v_1. Consider the sequence $v_1, v_2, \ldots, v_n$. We now give a greedy $L(2,1)$-coloring of the vertices of T with colors from the set S. Assign the color 0 to v_1 and the color 2 to v_2. Suppose now that an $L(2,1)$-coloring of the subtree T_i of T induced by $\{v_1, v_2, \ldots, v_i\}$ has been given, where $2 \leq i < n$. We assign v_{i+1} the smallest color from the set S so that an $L(2,1)$-coloring of the subtree T_{i+1} of T induced by $\{v_1, v_2, \ldots, v_{i+1}\}$ results. From the manner in which the sequence $v_1, v_2, \ldots, v_n$ was constructed, v_{i+1} is an end-vertex of T_{i+1} and so v_{i+1} is adjacent to exactly one vertex v_j with $1 \leq j \leq i$. The vertex v_j is adjacent to at most $\Delta - 1$ vertices in the subtree T_i. Hence v_{i+1} can be assigned a color that differs from those assigned to at most $\Delta - 1$ vertices and differs from any color within 1 of the color assigned to v_j. Hence at most $(\Delta - 1) + 3 = \Delta + 2$ colors cannot be used to color v_{i+1}, leaving at least one available color in S to color v_{i+1}. Thus $\lambda(T) \leq \Delta + 2$. ∎

By Theorem 14.5, $\lambda(K_{1,t}) = \Delta(K_{1,t}) + 1$ for every positive integer t. Thus $\lambda(P_2) = \Delta(P_2) + 1$ and $\lambda(P_3) = \Delta(P_3) + 1$. In addition, $\lambda(P_4) = \Delta(P_4) + 1$. For $n \geq 5$, however,

$$\lambda(P_n) = \Delta(P_n) + 2 = 4,$$

as we now show. Let $P_n = (v_1, v_2, \ldots, v_n)$. Consider the subgraph of P_n induced by the vertices v_i $(1 \leq i \leq 5)$, namely $P_5 = (v_1, v_2, v_3, v_4, v_5)$. The $L(2,1)$-coloring of P_5 given in Figure 14.6 shows that $\lambda(P_5) \leq 4$.

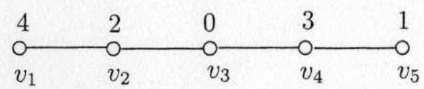

Figure 14.6: An $L(2,1)$-coloring of P_5

Since $\lambda(P_4) = 3$, it follows that $\lambda(P_5) \geq 3$. Suppose that $\lambda(P_5) = 3$. Then there is an $L(2,1)$-coloring c of P_5 using the colors 0, 1, 2, 3. Either c or $\overline{c}$ assigns the color 0 or 1 to v_3. Suppose that c assigns 0 or 1 to v_3. If $c(v_3) = 0$, then we may assume that $c(v_2) = 2$ and $c(v_4) = 3$. Then $c(v_1) = 0$, which is impossible. Hence $c(v_3) = 1$. However then, at most one of v_1 and v_4 is colored 3, which is impossible. Therefore, $\lambda(P_5) = 4$, which implies by Theorem 14.6 that $\lambda(P_n) = 4$ for $n \geq 5$.

By Theorem 14.6, $\Delta + 1 \leq \lambda(T) \leq \Delta + 2$ for every tree T with maximum degree Δ. If T has order n, then $\Delta \leq n - 1$ and so $\lambda(T) \leq (n-1) + 2 = n + 1$ for every tree T of order n. However, if $\Delta = n - 1$, then T is a star and $\lambda(T) = \Delta + 1 \leq n$. Therefore, for every tree T of order n, $\lambda(T) \leq n$. In fact, $\lambda(G) \leq n$ for every bipartite graph G of order n, which follows from a more general upper bound of Griggs and Yeh [83] for the L-span of a graph.

Theorem 14.7 *If G is a graph of order n, then*

$$\lambda(G) \leq n + \chi(G) - 2.$$

Proof. Suppose that $\chi(G) = k$. Then $V(G)$ can be partitioned into k independent sets $V_1, V_2, \ldots, V_k$, where $|V_i| = n_i$ for $1 \leq i \leq k$. Assign the colors $0, 1, 2, \ldots, n_1 - 1$ to the vertices of V_1 and for $2 \leq i \leq k$, assign the colors

$$\begin{gathered} n_1 + n_2 + \cdots + n_{i-1} + (i-1), \\ n_1 + n_2 + \cdots + n_{i-1} + i, \\ \vdots \\ n_1 + n_2 + \cdots + n_i + (i-2), \end{gathered}$$

to the vertices of V_i. Since this is an $L(2,1)$-coloring of G, it follows that

$$\lambda(G) \leq n + k - 2,$$

as desired. ■

An immediate consequence of Theorem 14.7 is the following.

Corollary 14.8 *If G is a complete k-partite graph of order n, where $k \geq 2$, then*

$$\lambda(G) = n + k - 2.$$

Proof. Let G be a complete k-partite graph with partite sets $V_1, V_2, \ldots, V_k$. By Theorem 14.7, $\lambda(G) \leq n + k - 2$. Let c be an $L(2,1)$-coloring of G with c-span $\lambda(G)$ using colors from the set $S = \{0, 1, \ldots, \lambda(G)\}$ and let a_i be the largest color assigned to a vertex of V_i $(1 \leq i \leq k)$. Since every two distinct vertices of G are either adjacent or at distance 2, it follows that c must assign distinct colors to all n vertices of G. Furthermore, since every two vertices of G belonging to different partite sets are adjacent, it follows that no vertex of G can be colored $a_i + 1$ for any i $(1 \leq i \leq k)$. Hence there are $k - 1$ colors of S that cannot be assigned to any vertex of G, which implies that the largest color that c can assign to a vertex of G is at least $(n-1) + (k-1) = n + k - 2$ and so $\lambda(G) \geq n + k - 2$. Therefore, $\lambda(G) = n + k - 2$. ■

While we have already noted that $\lambda(G) \geq \Delta + 1$ for every graph G with maximum degree Δ, many of the upper bounds for $\lambda(G)$ have also been expressed in terms of Δ. For example, Griggs and Yeh [83] obtained the following.

Theorem 14.9 *If G is a graph with maximum degree Δ, then*

$$\lambda(G) \leq \Delta^2 + 2\Delta.$$

Proof. For a given sequence $v_1, v_2, \ldots, v_n$ of the vertices of G, we now conduct a greedy $L(2,1)$-coloring c of G. We begin by defining $c(v_1) = 0$. For each vertex v_i $(2 \leq i \leq n)$, at most Δ vertices of G are adjacent to v_i and at most $\Delta^2 - \Delta$ vertices of G are at distance 2 from v_i. Hence when assigning a color to v_i, if a vertex v_j

adjacent to v_i precedes v_i in the sequence, then we must avoid assigning v_i any of the three colors $c(v_j) - 1$, $c(v_j)$, $c(v_j) + 1$; while if a vertex v_j is at distance 2 from v_i and precedes v_i in the sequence, then we must avoid assigning v_i the color $c(v_j)$. Therefore, there are at most $3\Delta + (\Delta^2 - \Delta) = \Delta^2 + 2\Delta$ colors to be avoided when coloring any vertex v_i $(2 \leq i \leq n)$. Hence at least one of the $\Delta^2 + 2\Delta + 1$ colors $0, 1, 2, \ldots, \Delta^2 + 2\Delta$ is available for v_i and so $\lambda(G) \leq \Delta^2 + 2\Delta$. ■

Griggs and Yeh [83] also showed that if if a graph G has diameter 2, then the bound $\Delta^2 + 2\Delta$ for $\lambda(G)$ in Theorem 14.9 can be improved.

Theorem 14.10 *If G is a connected graph of diameter 2 with $\Delta(G) = \Delta$, then*

$$\lambda(G) \leq \Delta^2.$$

Proof. If $\Delta = 2$, then G is either P_3, C_4, or C_5. The $L(2,1)$-colorings of these three graphs in Figure 14.7 show that $\lambda(G) \leq 4$ for each such graph G. Hence we can now assume that $\Delta \geq 3$. Suppose that the order of G is n. We consider two cases for Δ, according to whether Δ is large or small in comparison with n.

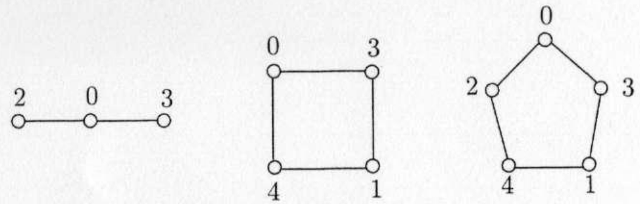

Figure 14.7: $L(2,1)$-colorings of the three graphs G with $\Delta(G) = \text{diam}(G) = 2$

Case 1. $\Delta \geq (n-1)/2$. Since G is neither a cycle nor a complete graph, it follows from Brooks' theorem (Theorem 7.12) that $\chi(G) \leq \Delta$. By Theorem 14.7,

$$\begin{aligned} \lambda(G) &\leq n + \chi(G) - 2 \leq (2\Delta + 1) + \Delta - 2 \\ &= 3\Delta - 1 < \Delta^2, \end{aligned}$$

the final inequality follows because $\Delta \geq 3$.

Case 2. $\Delta \leq (n-2)/2$. Therefore, $\delta(\overline{G}) \geq n/2$. By Corollary 3.8, $\overline{G}$ is Hamiltonian and so contains a Hamiltonian path $P = (v_1, v_2, \ldots, v_n)$. Define a coloring c on G by $c(v_i) = i - 1$ for $1 \leq i \leq n$. Since every two vertices of G with consecutive colors are adjacent in $\overline{G}$, these vertices are not adjacent in G. Thus c is an $L(2,1)$-coloring of G and the c-span is $n - 1$, which implies that $\lambda(G) \leq n - 1$.

Now, for each vertex v of G, at most Δ vertices are adjacent to v and at most $\Delta^2 - \Delta$ vertices are at distance 2 from v. Since the diameter of G is 2, all vertices of G are within distance 2 of v and so

$$n \leq 1 + \Delta + (\Delta^2 - \Delta) = \Delta^2 + 1.$$

Therefore, $\lambda(G) \leq n-1 \leq \Delta^2$. ∎

The proof of the preceding theorem shows that for a connected graph G of order n, diameter 2, and maximum degree Δ, the bound Δ^2 for $\lambda(G)$ can only be attained when $\Delta = 2$ (which occurs for C_4 and C_5) or when $\Delta \geq 3$ and $n = \Delta^2+1$, which can only occur, by a theorem due to Alan Hoffman and Robert Singleton [106], when $\Delta = 3$ or $\Delta = 7$, or possibly when $\Delta = 57$. When $\Delta = 3$, there is only one such graph, namely the Petersen graph (see Exercise 12). When $\Delta = 7$, there is also only one such graph, called the **Hoffman-Singleton graph**. When $\Delta \notin \{2,3,7\}$, it is known that there is no graph of diameter 2 and Δ^2+1 except possibly when $\Delta = 57$ (see [106]). The mysterious situation surrounding the existence or non-existence of a graph of diameter 2, maximum degree 57, and order 57^2+1 has never been resolved. Because $\text{diam}(G) = 2$, every $L(2,1)$-coloring of G must assign distinct colors to the vertices of G and so $\lambda(G) \geq n-1 = \Delta^2$. However, by Theorem 14.10, $\lambda(G) \leq \Delta^2$. Thus $\lambda(G)$ can equal Δ^2 only when $\Delta \in \{2,3,7\}$ or possibly when $\Delta = 57$.

Griggs and Yeh [83] also described a class of graphs G with maximum degree Δ for which $\lambda(G) = \Delta^2 - \Delta$. These are the incidence graphs of finite projective planes. A **finite projective plane** of order $n \geq 2$ is a set of n^2+n+1 objects called points and a set of n^2+n+1 objects called lines such that each point is incident with (lies on) $n+1$ lines and each line is incident with (contains) $n+1$ points. It is known that if n is a power of a prime, then a projective plane of order n exists. In particular, there is a projective plane of order 2 (containing $2^2+2+1 = 7$ points and 7 lines) and a projective plane of order 3 (containing 13 points and 13 lines). The **incidence graph of a projective plane** of order n is a bipartite graph G with partite sets V_1 and V_2 , where V_1 is the set of points and V_2 is the set of lines and uv is an edge of G if one of u and v is a point and the other is a line incident with this point. Thus $|V_1| = |V_2| = n^2+n+1$ and so G is an $(n+1)$-regular bipartite graph of order $2(n^2+n+1)$. In the simplest case, the projective plane of order 2 (also called the **Fano plane**) is a 3-regular graph of order 14. In this case, the set of points can be denoted by

$$V_1 = \{1,2,3,4,5,6,7\}$$

and the set of lines by

$$V_2 = \{(123),(246),(145),(257),(347),(356),(167)\}.$$

The incidence graph of this projective plane is shown in Figure 14.8. This graph is called the **Heawood graph** and is a cubic graph of smallest order (namely 14) having girth 6.

In the incidence graph G of a projective plane of order n, the distance between every two vertices of V_i $(i = 1,2)$ is 2 and the distance between two nonadjacent vertices belonging to different partite sets is 3. Consequently, no two vertices of V_1 or of V_2 can be assigned the same color in an $L(2,1)$-coloring of G. This says that $\lambda(G) \geq n^2+n$. Because there is an $L(2,1)$-coloring of G using the colors $0,1,\ldots,n^2+n$, it follows that $\lambda(G) \leq n^2+n$ and so $\lambda(G) = n^2+n$. Since in this case $\Delta^2 - \Delta = (n+1)^2-(n+1) = n^2+n$, we have $\lambda(G) = \Delta^2 - \Delta$. Therefore,

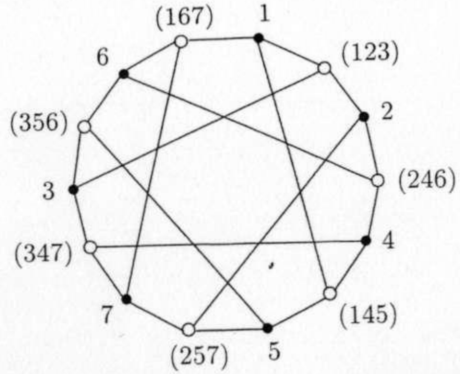

Figure 14.8: The incidence graph of the projective plane of order 2

the L-span of the incidence graph G of the projective plane of order 2 is 6. An $L(2,1)$-coloring of this graph using the colors $0,1,\ldots,6$ is shown in Figure 14.9.

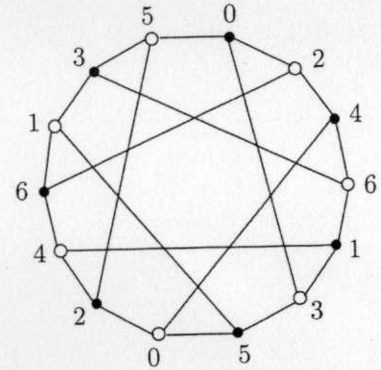

Figure 14.9: An $L(2,1)$-coloring of the incidence graph of the projective plane of order 2

In the proof of Theorem 14.10 it was shown that if $\Delta \geq 3$ and $\Delta \geq (n-1)/2$, then $\lambda(G) \leq \Delta^2$. This particular argument did not make use of the assumption that G has diameter 2. This led Griggs and Yeh [83] to make the following conjecture.

Conjecture 14.11 *If G is a graph with $\Delta(G) = \Delta \geq 2$, then $\lambda(G) \leq \Delta^2$.*

In 2008 Frédéric Havet, Bruce Reed, and Jean-Sébastien Sereni [98] established the following.

Theorem 14.12 *There exists a positive integer N such that for every graph G of maximum degree $\Delta \geq N$,*

$$\lambda(G) \leq \Delta^2.$$