Advanced Parallel Architecture
Lesson 9

Annalisa Massini - 2016/2017

Residue number systems
Circuit metrics: area and delay

Residue number systems

- Residue number systems are based on the *congruence* relation:
  - Two integers a and b are said to be ***congruent modulo m*** if m divides exactly the difference of a and b
  - We write $a \equiv b \pmod{m}$
- For example
  - $10 \equiv 7 \pmod{3}$
  - $10 \equiv 4 \pmod{3}$
  - $10 \equiv 1 \pmod{3}$
  - $10 \equiv -2 \pmod{3}$
- The number m is a *modulus* or *base*, and we assume that its values exclude 1, which produces only trivial congruences

Residue number systems

- Infact:
- If ***q*** and ***r*** are the **quotient** and **remainder**, respectively, of the integer division of a by m - that is: $a = q{:}m + r$

  → then, by definition, we have $a \equiv r \pmod{m}$

- The number ***r*** is said to be the ***residue* of *a* with respect to *m***, and we shall usually denote this by $r = |a|_m$
- The set of m smallest values, $\{0; 1; 2; \ldots ; m - 1\}$, that the residue may assume is called the set of *least positive residues modulo m*

Residue number systems

- Suppose we have a set, $\{m_1; m_2; \ldots; m_N\}$, of N positive and pairwise **relatively prime** moduli
- Let M be the product of the moduli $M = m_1 \times m_2 \times \ldots \times m_N$
- We write the representation in the form $\langle x1; x2; \ldots; xN \rangle$, where $xi = |X|_{mi}$, and we indicate the relationship between X and its residues by writing $X \approx \langle x1; x2; \ldots; xN \rangle$
- Example: in the residue system {2, 3, 5}, M=30 and

$$8 \rightarrow \langle 0, 2, 3 \rangle$$

$$16 \rightarrow \langle 0, 1, 1 \rangle$$

Residue number systems

- Every number $X < M$ has a **unique representation** in the residue number system, which is the sequence of residues $<|X|_{mi} : 1 \le i \le N>$
- A partial proof of uniqueness is as follows:
  - Suppose X_1 and X_2 are two different numbers with the **same *residue representation***
  - Then $|X_1|_{mi} = |X_2|_{mi}$, and so $|X_1 - X_2|_{mi} = 0$
  - Therefore $X_1 - X_2$ is the least common multiple (**lcm**) of *mi*
  - But if the *mi* are relatively prime, then their **lcm** is *M*, and it must be that $X_1 - X_2$ is a multiple of *M*
  - So it cannot be that $X_1 < M$ and $X_2 < M$
  - Therefore, the representation $<|X|_{mi} : 1 \le i \le N>$ is unique and may be taken as the representation of *X*

Residue number systems

- The number M is called the *dynamic range* of the RNS, because the number of numbers that can be represented is M
- For unsigned numbers, that range is $[0;M - 1]$
- Representations in a system in which the moduli are not pairwise relatively prime will be not be unique: two or more numbers will have the same representation

- Example

	Relatively prime			Relatively non-prime		
N	**m1=2**	**m2=3**	**m3=5**	**m1=2**	**m2=4**	**m3=6**
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	0	2	2	0	2	2
3	1	0	3	1	3	3
4	0	1	4	0	0	4
5	1	2	0	1	1	5
6	0	0	1	0	2	0
7	1	1	2	1	3	1
8	0	2	3	0	0	2
9	1	0	4	1	1	3
10	0	1	0	0	2	4
11	1	2	1	1	3	5
12	0	0	2	0	0	0
13	1	1	3	1	1	1
14	0	2	4	0	2	2
15	1	0	0	1	3	3

Residue number systems

- We defined *standard residue number systems*
- There are also examples of *non-standard* RNS, the most common of which are the *redundant residue number systems*
- Such a system is obtained by, essentially, adding extra (redundant) moduli to a standard system
- The dynamic range then consists of a **legitimate** range, defined by the non-redundant moduli and an **illegitimate** range
- Redundant number systems of this type are especially useful in fault-tolerant computing

Residue number systems

- Ignoring other, more *practical*, issues, the best moduli are probably **prime numbers**

- For computer applications, it is important to have moduli-sets that facilitate both efficient representation and balance, meaning that the *differences between the moduli should be as small as possible*

Residue number systems

- Take, for example, the choice of 13 and 17 for the moduli that are adjacent prime numbers

- The dynamic range is 221

- With a straightforward binary encoding:
  - 4 bits will be required to represent 13
  - 5 bits will be required to represent 17

Residue number systems

- The representational efficiency is:
  - In the first case 13/16
  - In the second case is 17/32

- If instead we chose 13 and 16, then the representational efficiency:
  - is improved to 16/16 in the second case
  - but at the cost of **reduction in the range** (down to 208)
- With the better balanced pair, 15 and 16, we would have:
  - a better efficiency 15/16 and 16/16
  - A greater range: 240

Residue number systems

- It is also useful to have ***moduli that simplify*** the implementation of the ***arithmetic operations***

- This means that arithmetic on residue digits should ***not deviate too far from conventional arithmetic***, which is just **arithmetic modulo a power of two**

- A common choice of prime modulus that does not complicate arithmetic and which has good representational efficiency is $m_i = 2^i - 1$

Residue number systems

- Not all pairs of numbers of the form $2^i - 1$ are relatively prime
- It can be shown that that $2^j - 1$ and $2^k - 1$ are relatively prime **if and only if** *j* and *k* are relatively prime

- For example:
  - $2^4-1= 15$ 15=3x5
  - $2^5-1= 31$ 31 *prime*
  - $2^6-1= 63$ 63=3x7
  - $2^7-1= 127$ 127 *prime*
  - $2^8-1= 255$ 255=3x5x17

Residue number systems

- Many moduli sets are based on these choices, but there are other possibilities; for example, moduli-sets of the form $\{2^n-1; 2^n; 2^n+1\}$ are among the most popular in use
- At least four considerations for the selection of moduli
  - The selected moduli must provide an **adequate range** whilst also ensuring that RNS representations are **unique**
  - The **efficiency of binary representations**; a balance between the different moduli in a given moduli-set is also important
  - The **implementations of arithmetic units** for RNS should to some extent be compatible with those for conventional arithmetic, especially given the legacy that exists for the latter
  - The **size of individual moduli**

Residue number systems

- One of the primary **advantages** of RNS is that certain RNS-arithmetic operations do not require carries between digits
- But, this is so only between *digits*
- Since a digit is ultimately represented in binary, there will be carries between bits, and therefore it is important to ensure that digits (→ the moduli) are **not too large**

Residue number systems

- Small digits make it possible to realize cost-effective table-lookup implementations of arithmetic operations

- But, on the other hand, if the moduli are small, then a large number of them may be required to ensure a sufficient dynamic range

- The choices depend on applications and technologies

Residue number systems

Negative numbers

- As with the conventional number systems, any one of the radix complement, diminished-radix complement, or sign-and-magnitude notations may be used in RNS
- The merits and drawbacks of choosing one over the other are similar to those for the conventional notations
- However, the **determination of sign** is much *more difficult* with the residue notations, as is **magnitude-comparison**
- This problem imposes many limitations on the application of RNS and we deal with just the positive numbers

Residue number systems

Basic arithmetic

- Addition/subtraction and multiplication are easily implemented with residue notation, depending on the choice of the moduli

- Division is much more difficult due to the difficulties of sign-determination and magnitude-comparison

Residue number systems

Basic arithmetic

- Residue **addition** is carried out by individually adding corresponding digits
- A **carry**-out from one digit position is **not propagated** into the next digit position
- As an example, with the moduli-set {2; 3; 5; 7}:
  - the representation of 17 is <1; 2; 2; 3>
  - the representation of 19 is <1; 1; 4; 5>
  - adding the two residue numbers yields <0; 0; 1; 1>, which is the representation for 36 in that system

Residue number systems

Basic arithmetic

- **Subtraction** may be carried out by negating (in whatever is the chosen notation) the subtrahend and adding to the minuend
- This is straightforward for numbers in diminished-radix complement or radix complement notation
- For sign-and-magnitude representation, a slight modification of the algorithm for conventional sign-and-magnitude is necessary:
  - the sign digit is fanned out to all positions
  - addition proceeds as in the case for unsigned numbers but with a conventional sign-and-magnitude algorithm.

Residue number systems

Basic arithmetic

- **Multiplication** too can be performed simply by multiplying corresponding residue digit-pairs, relative to the modulus for their position → multiply digits and ignore or adjust an appropriate part of the result
- As an example, with the moduli-set {2; 3; 5; 7}:
  - 17 → <1; 2; 2; 3>
  - 19 → <1; 1; 4; 5>
  - their product, 323 is <1; 2; 3; 1>

Residue number systems

Basic arithmetic

- Basic fixed-point division consists, essentially, of a sequence of subtractions, magnitude-comparisons, and selections of the quotient-digits
- But **comparison** in RNS is a diffcult operation, because RNS is not positional or weighted
- Example:
  - moduli-set {2; 3; 5; 7}
  - the number represented by <0; 0; 1; 1> is almost twice that represented by <1; 1; 4; 5>
  - but this is far from apparent

Residue number systems

Conversion

- The most direct way to convert from a conventional representation to a residue one is to divide by each of the given moduli and then collect the remainders, *forward conversion*
- This is a **costly** operation if the number is represented in an **arbitrary radix** and the **moduli are arbitrary**
- If number is represented in **radix-2** (or a radix that is a power of two) and the moduli are of a suitable form (e.g. 2^n-1), then these procedures that can be implemented with more efficiency
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Residue number systems

Conversion

- The conversion from residue notation to a conventional notation - ***reverse conversion*** - is more difficult (conceptually, if not necessarily in the implementation) and so far has been one of the major impediments to the adoption use of RNS
  - One way in which it can be done is to assign weights to the digits of a residue representation and then produce a positional (weighted) mixed-radix representation that can then be converted into any conventional form
  - Another approach involves the use of the Chinese Remainder Theorem, which is the basis for many algorithms for conversion from residue to conventional notation

Residue number systems

Base extension

- A frequently occurring computation is that of *base extension*, which is defined as:
  - Given a residue representation $<|X|_{m1} ; |X|_{m2} ; \ldots ; |X|_{mN}>$ and an additional set of moduli, $m_{N+1};m_{N+2}; \ldots ;m_{N+K}$, such that $m_1;m_2; \ldots m_N;m_{N+1}; \ldots ;m_{N+K}$ are all pairwise relatively prime
  - we want to compute the residue representation $<|X|_{m1} ; |X|_{m2} ; \ldots ; |X|_{mN}; |X|_{mN+1}, \ldots ; |X|_{mN+K}>$
- Base extension is useful in dealing with the diffcult operations of reverse conversion, division, dynamic-range extension, magnitude-comparison, overflow-detection, and sign-determination

Residue number systems

- **Example**: *multiply-accumulate operation* over a sequence of scalars (frequent operation in digital-signal processing)
  - Let the moduli-set be {2; 3; 5; 7} with dynamic range 210
  - We wish to evaluate the sum-of-products 7 x 3 + 16 x 5 + 47 x 2
  - The residue-sets are
    - 2 → <0; 2; 2; 2>
    - 3 → <1; 0; 3; 3>
    - 5 → <1; 2; 0; 5>
    - 7 → <1; 1; 2; 0>
    - 16 → <0; 1; 1; 2>
    - 47 → <1; 2; 2; 5>

Residue number systems

- **Example**: *multiply-accumulate operation* over a sequence of scalars (frequent operation in digital-signal processing)
- We proceed by first computing the products by multiplying the corresponding residues:
  - $7 \text{ x } 3 \rightarrow < |1\text{x}1|_2 \; |1\text{x}0|_3 \; |2\text{x}3|_5 \; |0\text{x}3|_7 > = < 1, 0, 1, 0>$
  - $16 \text{ x } 5 \rightarrow < |0\text{x}1|_2 \; |1\text{x}2|_3 \; |1\text{x}0|_5 \; |2\text{x}5|_7 > = < 0, 2, 0, 3>$
  - $47 \text{ x } 2 \rightarrow < |1\text{x}0|_2 \; |2\text{x}2|_3 \; |2\text{x}2|_5 \; |5\text{x}2|_7 > = < 0, 1, 4, 3>$
- The sum of products can be evaluated by adding the corresponding residues:
  - $<|1+0+0|_2 \; |0+2+1|_3 \; |1+0+4|_5 \; |0+3+3|_7 > = < 1, 0, 0, 6>$

Circuit area and time evaluation

Circuit area and time

- To discuss about the time and area, it is useful the analytical model (unit-gate model) presented in
  - A. Tyagi, *A reduced-area scheme for carry-select adders*, IEEE Trans. Comput., 1993

- They use a simplistic model for gate-count and gate-delay:
  - Each gate except **EX-OR** counts as one elementary gate
  - An **EX-OR** gate is counted as two elementary gates, because in static (restoring) CMOS, an **EX-OR** gate is implemented as two elementary gates **(NAND)**
  - The delay through an elementary gate is counted as one gate-delay unit, but an **EX-OR** gate is two gate-delay units

Circuit area and time

- In this model we are ignoring the *fanin* and *fanout* of a gate
- This can lead to unfair comparisons for circuits containing gates with a large difference in *fanin* or *fanout*
  - For instance, gates in the CLA adder have different *fanin*
  - A carry-ripple adder has no gates with *fanin* and *fanout* greater than 2

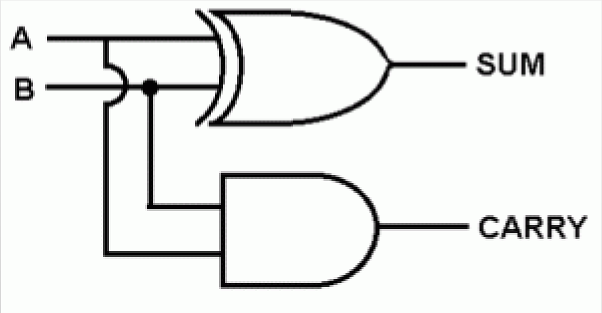
- The best comparison for a VLSI implementation is actual area and time
- The gate-count and gate-delay comparisons may not always be consistent with the area-time comparisons

Circuit area and time

- To simplify we consider:
  - **Any gate** (but the EX-OR) counts as **one gate** for both area and delay → A_{gate} and T_{gate}
  - An **exclusive-OR gate** counts as **two elementary gates** for both area and delay → $A_{EX\text{-}OR} = 2A_{gate}$ and $T_{EX\text{-}OR} = 2T_{gate}$
  - An ***m*-input gate** counts as ***m* – 1 gates for area** and **$\log_2 m$ gates** for delay → $A_{m\text{-}gate} = (m-1)A_{gate}$ and $T_{m\text{-}gate} = \log_2 m\ T_{gate}$

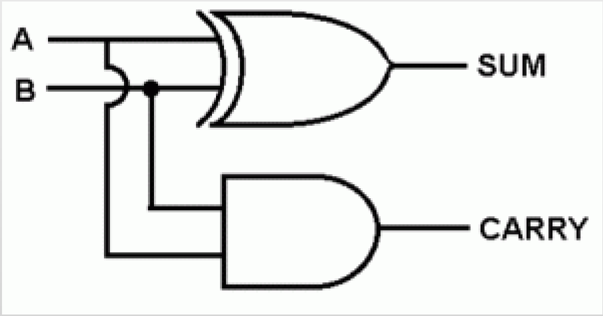
Circuit area and time

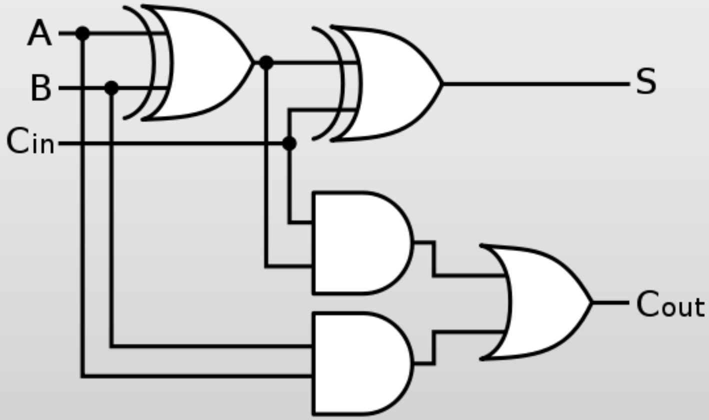
- A half adder (HA) has:
  - delay 2 unit gates → $T_{HA} = 2\ T_{gate}$
  - area 3 unit gates → $A_{HA} = 3\ A_{gate}$



Circuit area and time

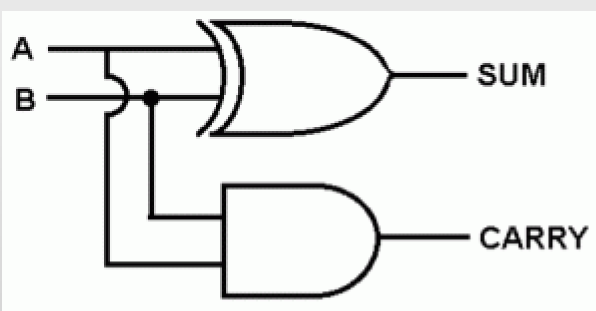
- A half adder (HA) has:
  - delay 2 unit gates → $T_{HA} = 2\ T_{gate}$
  - area 3 unit gates → $A_{HA} = 3\ A_{gate}$
- A full adder (FA) has:
  - delay 4 unit gates → $T_{FA} = 4\ T_{gate}$
  - area 7 unit gates → $A_{FA} = 7\ A_{gate}$

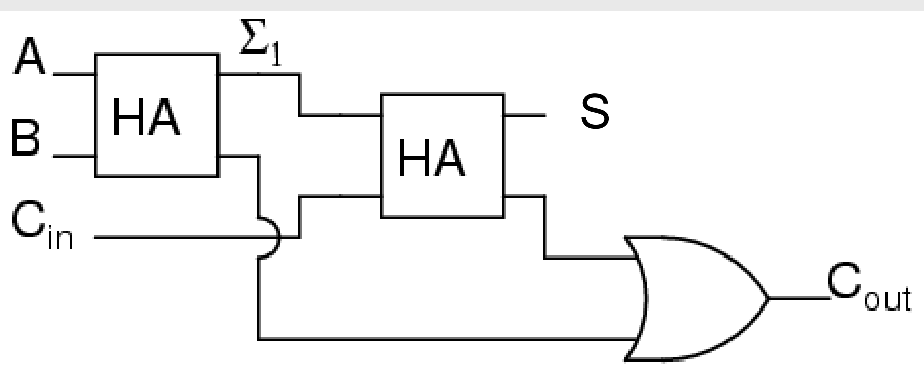




Circuit area and time

- A half adder (HA) has:
  - delay 2 unit gates → $T_{HA} = 2\ T_{gate}$
  - area 3 unit gates → $A_{HA} = 3\ A_{gate}$
- A full adder (FA) has:
  - delay 4 unit gates → $T_{FA} = 4\ T_{gate} = 2\ T_{HA}$
  - area 7 unit gates → $A_{FA} = 7\ A_{gate} = 2\ A_{HA} + A_{gate}$





Circuit area and time

- A carry-ripple adder for n-bits operands has:
  - delay $T_{CR\text{-}adder}$ → $T_{CR\text{-}adder} = n\ T_{FA} = 2n\ T_{HA} = 4n\ T_{gate}$
  - area $A_{CR\text{-}adder}$ → $A_{CR\text{-}adder} = n\ A_{FA} = 2n\ A_{HA} + n\ A_{gate} = 7n\ A_{gate}$